

NEAR PARABOLIC RENORMALIZATION FOR UNICRITICAL HOLOMORPHIC MAPS

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ABSTRACT. Inou and Shishikura provided a class of maps that is invariant by near-parabolic renormalization, and that has proved extremely useful in the study of the dynamics of quadratic polynomials. We provide here another construction, using more general arguments. This will allow to extend the range of applications to unicritical polynomials of all degrees.

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Notations: \mathbb{D} refers to the unit disk in the complex plane: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and \mathbb{H} to the upper half plane: $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. The translation by 1 in \mathbb{C} is denoted by $T_1 : z \mapsto z + 1$. By convention, \mathbb{N} includes 0 and we will denote \mathbb{N}^* the set of positive integers. Beyond its usual meaning as Archimedes’ constant, the

This research was partially funded by the Agence Nationale de la Recherche, Grant ABC (At the Boundary of Chaos) ARN-08-JCJC-0002-01.

symbol π will often refer to the canonical projection $\pi : \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$. We will often make use of the map $E(z) = e^{2i\pi z}$. We adopt the following convention for open and semi-open intervals: $]a, b[$, $[a, b[$, $]a, b]$. An upper half plane means a half plane (usually open) bounded by a horizontal line and sitting above it. The restriction of a map f to the set A is denoted $f|_A$. The floor of $x \in \mathbb{R}$, i.e. the greatest relative integer $n \in \mathbb{Z}$ such that $n \leq x$, is denoted $\lfloor x \rfloor$. The notation \mathcal{S} refers to the class of Schlicht maps, i.e. holomorphic injective maps $\phi : \mathbb{D} \rightarrow \mathbb{C}$ with $\phi(0) = 0$ and $\phi'(0) = 1$. There are a lot of more specific notations in this article, and a (partial) summary of symbols has been added near the end.

Conventions: The hyperbolic metric on \mathbb{D} is chosen to be $\frac{|dz|}{1-|z|^2}$, and the hyperbolic metric on open strict subsets U of \mathbb{C} is normalized according to this convention, i.e. it is the image of the metric of the disk by its identification with the universal cover of U . With that convention, the hyperbolic metric on \mathbb{H} takes the form $|dz|/2\text{Im } z$. (Some authors prefer using $\frac{2|dz|}{1-|z|^2}$ on \mathbb{D} so that one gets $|dz|/\text{Im } z$ on \mathbb{H} .)

1. INTRODUCTION

This article has a long introduction and the main theorem appears only on page 12.

1.1. Structural equivalence. In the breakthrough by Inou and Shishikura [IS04], they make use of a class of maps defined as follows (notations and details may differ): \mathcal{F}_{IS} is the set of maps of the form $f = P \circ \phi^{-1}$ where ϕ varies among the univalent maps on V on such that $\phi(z) = z + \mathcal{O}(z^2)$ at the origin. Here $P(z) = z(1+z)^2$ and V is a specific open subset of \mathbb{C} containing 0 defined in their article. The set \mathcal{F}_{IS} is better thought of as the set of maps that cover the plane in a specific way, and with $f(z) = z + \mathcal{O}(z^2)$. They are not covers because they have ramification points. And they are not even ramified covers, because the cardinality of the preimage of a point is not constant, even when counted with multiplicity. So they are a sort of partial ramified covers.¹ This class \mathcal{F}_{IS} comes in fact from another class of maps, invariant by parabolic renormalization (defined later in this section), with a much richer ramified cover structure, but which was too rigid for their purposes, which was to have a class invariant by *near* parabolic renormalization. They extracted a carefully chosen subset of this structure to define their class \mathcal{F}_{IS} .

We are going to use the same idea, but we will keep more of the original ramified cover structure. Let us formalize the notion of structure:

Definition 1. Let X_1, X_2, Y be dimension one analytic manifolds². Consider an index set I , and two collections of marked points $a_i \in X_1$ and $b_i \in X_2$ indexed by $i \in I$. Consider also two analytic maps which are nowhere locally constant $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$. We will say that the pairs (a, f_1) and (b, f_2) are structurally equivalent if there exists an analytic isomorphism $\phi : X_1 \rightarrow X_2$ such that $f_1 = f_2 \circ \phi$ and $b = \phi \circ a$ i.e. such that the following diagram commutes

$$\begin{array}{ccc}
 & I & \\
 a \swarrow & & \searrow b \\
 X_1 & \xrightarrow{\phi} & X_2 \\
 f_1 \searrow & & \swarrow f_2 \\
 & Y &
 \end{array}$$

¹A good metaphor is with a Roman toga.

²a.k.a. Riemann surfaces

i.e. such that ϕ sends the marked point a_i to b_i and such that it sends the fiber $f_1^{-1}(y)$ in the fiber $f_2^{-1}(y)$ for all $y \in Y$. Note that this requires that $f_2 \circ b = f_1 \circ a$. Structural equivalence is an equivalence relation, which depends on I and Y . To specify them, we will sometimes use the terminology (I, Y) -structurally equivalent or structurally equivalent over Y with marker I . The equivalence classes will be called structures (or (I, Y) -structures).

We could also call this a *marked analytic partial ramified cover equivalence class* but it would be a long name for a simply defined notion.

The restriction of partial covers (without losing marked points) induces a pre-order on structures as follows:

Definition 2. *With the same definition as above, but assuming ϕ analytic injective instead of analytic isomorphism (thus dropping the surjectivity assumption), we will say that the structure of (a, f_1) is a sub-structure of that of (b, f_2) : this is indeed independent of the choice of representatives in their equivalence classes. We will also say that (b, f_2) has at least the structure of (a, f_1) . It is equivalent to the following: (a, f_1) is structurally equivalent to (b, g_2) where g_2 is a restriction of f_2 to a set containing the image of b . In other words sub-structures of (b, f_2) are equivalence classes of restrictions of f_2 to open sets containing the marked points.*

This preorder is not always an order: for instance if $I = \emptyset$, and the sets $X_1 \subset \mathbb{C}$ defined by $\operatorname{Re}(z) > 0$ and X_2 defined by $\operatorname{Re}(z) > 1/2$ are both mapped to \mathbb{C}/\mathbb{Z} using the canonical projection from \mathbb{C} to the quotient, then each has at least the structure of the other (take $\phi_1(z) = z + 1$ and $\phi_2(z) = z$), while they are not equivalent.

However, on the subclass of structures with connected X and at least one marked point, this preorder is an order:

Proof. Assume each of (a, f_1) and (b, f_2) has at least the structure of the other and assume that both X_i are connected and $I \neq \emptyset$. Call $\phi_1 : X_1 \rightarrow X_2$ and $\phi_2 : X_2 \rightarrow X_1$ the two analytic injections. We have to prove that (a, f_1) is structurally equivalent to (b, f_2) . It is sufficient to prove that ϕ_1 and ϕ_2 are surjective. Call $\zeta = \phi_2 \circ \phi_1$. It is injective, satisfies $f_1 \circ \zeta = f_1$ and fixes the marked points of f_1 . The map f_1 being not locally constant at the marked points, each marked point has a neighborhood on which some iterate ζ^m of the map ζ is the identity, where m is the local degree of f_1 at the marked point. Since there is at least one marked point and since X_1 is connected, $\zeta^m = \operatorname{id}$ holds everywhere by analytic continuation. Hence ϕ_2 is surjective. The proof is analogous for ϕ_1 . \square

1.2. Parabolic points. The present section is given mainly to fix notations. The reader that does not already know the theory of parabolic fixed points of one dimensional holomorphic dynamical systems will have hard times understanding the article, we recommend learning it in any of the classic books introducing holomorphic dynamics, or in [Dou94, Zin97]. The article [BE02] is also instructive and very well illustrated.

Consider a holomorphic dynamical system, $f : \operatorname{Dom}(f) \subset X \rightarrow X$. Assume it contains a parabolic point of period one, rotation number 0 and with one attracting petal, i.e. in some chart f has expression $f(z) = z + a_2 z^2 + \mathcal{O}(z^3)$ with $a_2 \neq 0$. To this are associated attracting Fatou coordinates Φ_{attr} and repelling Fatou coordinates Φ_{rep} and the local conjugacy invariant called horn maps. Let us quickly recall what these are.

Petals and Fatou coordinates: There are various domains on which the Fatou coordinates are usually defined by different authors, but the following is certainly

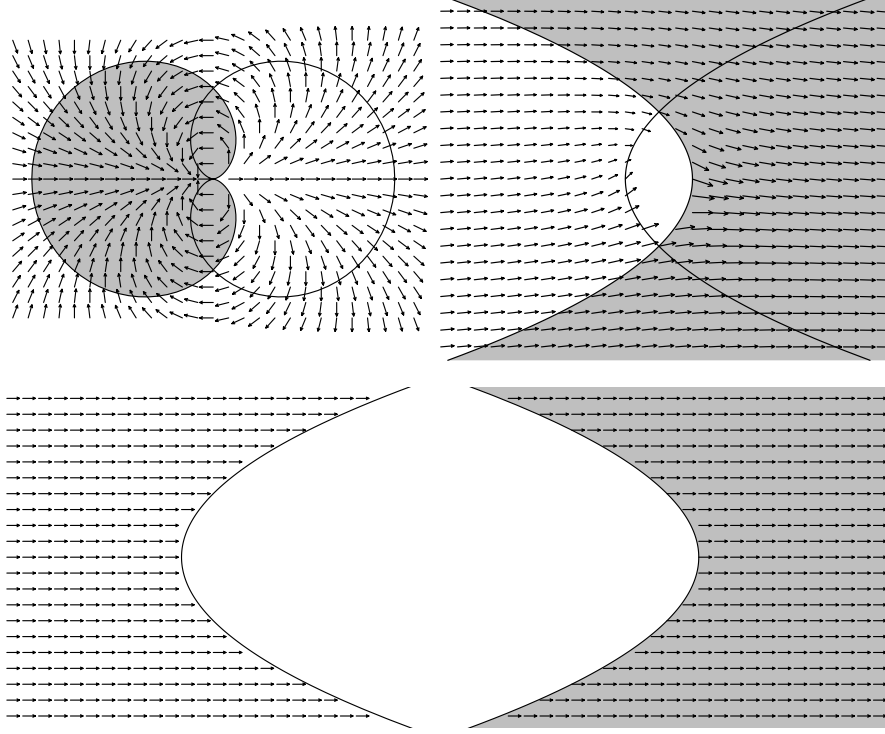


Figure 1: Illustration of petals and Fatou coordinates (schematic). Parabolic petals may look like the upper left picture. An attracting petal has been colored in gray. The arrows indicate the direction but not the length (for more readability) of the vector $f(z) - z$: the latter gets quite small near the fixed point. Upper right: The map $z \mapsto -1/a_2z$ will send the petals to the two regions bounded by parabolic-like curves (and exterior to these curves). Lower left and right: the repelling and attracting Fatou coordinates conjugate f to a translation and map both petals to two other regions that look very much like the upper right picture, but cannot be drawn on a same complex plane in a compatible way: this is precisely the horn maps that tell how they glue.

true: there exists a domain $\mathcal{P}_{\text{attr}}$ that we will call *(wide) attracting petal* and a function $\Phi_{\text{attr}} : \mathcal{P}_{\text{attr}} \rightarrow \mathbb{C}$ called *attracting Fatou coordinate* such that

- $\mathcal{P}_{\text{attr}}$ is open, non-empty, connected and simply connected,
- $f(\mathcal{P}_{\text{attr}}) \subset \mathcal{P}_{\text{attr}}$,
- $\forall z \in \mathcal{P}_{\text{attr}}, f^n(z) \rightarrow 0$ as $n \rightarrow +\infty$,
- conversely all orbits tending to 0 eventually fall in $\mathcal{P}_{\text{attr}}$ or on 0,
- Φ_{attr} is injective on $\mathcal{P}_{\text{attr}}$,
- (wide) its image $\Phi_{\text{attr}}(\mathcal{P}_{\text{attr}})$ contains big sectors as follows: $\forall \varepsilon, \exists R > 0$, $\{z \in \mathbb{C} \mid |z| > R \text{ and } |\arg(z)| < \pi - \varepsilon\} \subset \Phi_{\text{attr}}(\mathcal{P}_{\text{attr}})$,
- the following form of conjugacy holds:
 $\forall z \in \mathcal{P}_{\text{attr}}, \Phi_{\text{attr}} \circ f(z) = T_1 \circ \Phi_{\text{attr}}(z)$ with $T_1(z) = z + 1$,
- $\Phi_{\text{attr}}(z) \sim -1/a_2z$ as $z \rightarrow 0$.

The repelling version of the Fatou coordinate is similar to the attracting version for a branch of f^{-1} fixing the origin, with the difference that we ask Φ_{rep} to be the composition of an attracting Fatou coordinate for f^{-1} followed by $z \mapsto -z$, so that it still conjugates f to the translation by 1. Also $\Phi_{\text{rep}}(z) \sim -1/a_2z$ as $z \rightarrow 0$,

exactly as Φ_{attr} . The inverses $\Phi_{\text{attr}}^{-1}(z)$ and $\Phi_{\text{rep}}^{-1}(z)$ also satisfy this equivalent, but as $z \rightarrow \infty$. See Figure 1 for an illustration. The petals are not canonically defined: many variants exist satisfying the above conditions, many others exist satisfying other conditions, and it is not clear which definition should be preferred.

Normalization: For all $c, c' \in \mathbb{C}$ the maps $\Phi_{\text{attr}} + c$ and $\Phi_{\text{rep}} + c'$ satisfy the same properties. Conversely there is a form of uniqueness: Assume U_1 and U_2 are open sets, contained in $\text{Dom}(f)$, $f(U_i) \subset U_i$, all points in U_i have their orbit tending to 0 and every orbit tending to 0 is eventually either equal to 0 or contained in U_i . Assume that there are holomorphic functions (not assumed injective) $\Phi_i : U_i \rightarrow \mathbb{C}$ such that $\Phi_i(f(z)) = \Phi_i(z) + 1$ holds on U_i . Then $U_1 \cap U_2$ satisfies the same assumptions, in particular it is non-empty, and there exists a constant $c \in \mathbb{C}$ such that $\Phi_2(z) = \Phi_1(z) + c$ holds on $U_1 \cap U_2$. In particular, if one takes $\Phi_2 = \Phi_{\text{attr}}$ and $U_2 = \mathcal{P}_{\text{attr}}$, we see that Φ_1 must be equal to $c + \Phi_{\text{attr}}$ on the non-empty set $U_1 \cap \mathcal{P}_{\text{attr}}$. A similar statement holds for the repelling Fatou coordinate. So in some sense the Fatou coordinates are *unique up to addition of a constant*. So Fatou coordinates come in classes parameterized by a complex number. The choice of an element in a class is called a *normalization*.

Extension: There exists a unique extension of the attracting Fatou coordinate Φ_{attr} on the basin of the parabolic point, such that

$$\Phi_{\text{attr}} \circ f = T_1 \circ \Phi_{\text{attr}}.$$

Here, we mean in particular that the two compositions have the same domain of definition, which is possible iff the domain of Φ_{attr} is the whole basin of f . It can be defined as follows: let $\mathcal{P}_{\text{attr}}$ be an attracting petal on which a Fatou coordinate Φ is defined. For all z such that there exists $n \in \mathbb{N}$ with $f^n(z) \in \mathcal{P}_{\text{attr}}$, the quantity $\Phi(f^n(z)) - n$ is independent of n and we define $\Phi_{\text{attr}}(z)$ to be this complex number. The extended attracting Fatou coordinate plays the role of a greatest element in the set of attracting coordinates³. It is not necessarily injective. In the cases we will look at, it will not be. If so, the relation above is not a conjugacy but a semi-conjugacy.

There is no similar maximal element for the repelling Fatou coordinates. Instead, there exists a unique extension of the reciprocal $\Psi_{\text{rep}} = \Phi_{\text{rep}}^{-1}$ such that

$$\Psi_{\text{rep}} \circ T_1 = f \circ \Psi_{\text{rep}}.$$

Again, we want the domains of both compositions to be equal. The definition is similar: let \mathcal{P}_{rep} be a repelling petal. For all $z \in \mathbb{C}$, there exists $n \in \mathbb{N}$ such that $z - n \in \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$. The existence and the value of the quantity $f^n(\Phi_{\text{rep}}^{-1}(z - n))$ is independent of $n \geq 0$, and this defines $\Psi_{\text{rep}}(z)$. It is again holomorphic and not necessarily injective.

If f is a global map (a map whose orbits are all defined for all times, like a polynomial, an entire map, a rational map, ...) then Ψ_{rep} is defined on the whole complex plane \mathbb{C} .

Extended horn maps and parabolic renormalization: The *extended horn map* is the composition

$$h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$$

³For this to be correct we in fact set up an order relation on classes of Fatou coordinates, where equivalence is up to addition of a constant, define the order as inclusion of the domain of definition, and define Fatou coordinates as maps satisfying the weak assumptions given in the Normalization paragraph: i.e. we have at least to drop the injectivity assumption, as the greatest element will usually not satisfy it, and may also drop the big sectors assumption, though it is one that the greatest element does satisfy.

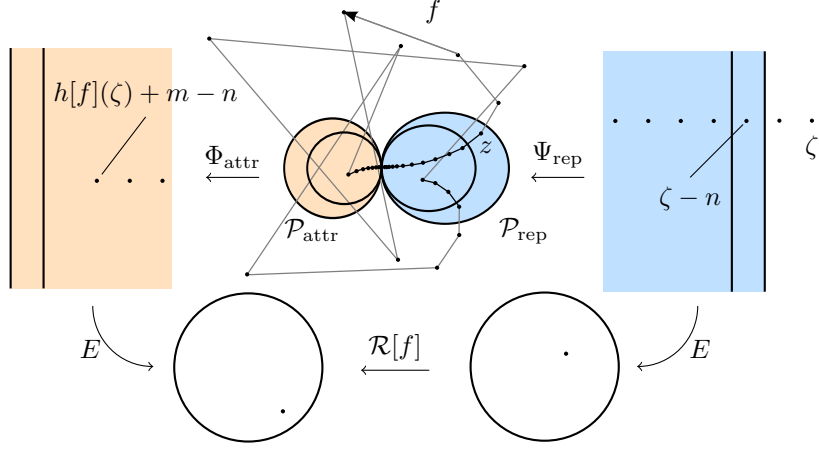


Figure 2: Decomposing $h[f]$ and $\mathcal{R}[f]$. For convenience, we have chosen petals $\mathcal{P}_{\text{attr}}$ and \mathcal{P}_{rep} whose image in Fatou coordinates are right and left half planes. Note that the orbit may visit the repelling petal more than one time, and does not necessarily enter the attracting petal by its leftmost part (the crescent shaped fundamental domain).

of these extensions. Changing the normalizations of the Fatou coordinates replaces h with its pre composition and post composition with two unrelated translations.

To define a renormalization, we proceed as follows. This definition does not pretend to be the best one, it is well suited to our purposes. The map h commutes with T_1 and its domain of definition is T_1 -invariant and contains an upper and a lower half plane. There is thus a quotient map $\text{Dom}(h)/\mathbb{Z} \rightarrow \mathbb{C}/\mathbb{Z}$. Conjugate it by $E : z \mapsto e^{2i\pi z}$ to a map defined on an open subset of \mathbb{C}^* containing a neighborhood of 0 and ∞ . With the properties of Fatou coordinates one proves that it can be continuously (and thus holomorphically) extended at these points, and that the extension fixes 0 and ∞ . For the *upper parabolic renormalization* of f , consider the restriction of this extension to the connected component of its domain of definition that contains 0, and possibly pre and post compose it with two linear maps ($z \mapsto az$ and $z \mapsto bz$) to be chosen according to conventions. For the lower parabolic renormalization of f , conjugate first the extension by $z \mapsto 1/z$, then restrict it to the connected component of the domain of definition containing 0 and finally compose with linear maps. The reason why we allow for these linear maps is that we will find it convenient later to use a different normalization for parabolic renormalization than for Fatou coordinates and the associated horn map.

Another point of view on extended horn maps, and parabolic renormalization: Since Φ_{attr} and Ψ_{rep} are defined beyond the petal $\mathcal{P}_{\text{attr}}$ and beyond $\Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$ by using iteration of f , the definition of $h[f]$ can be reformulated as follows:

- for $\zeta \in \text{Dom}(h[f])$, there exists $n \in \mathbb{N}$ such that $\zeta - n \in \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$,
- $\zeta - n = \Phi_{\text{rep}}(z)$ for a unique $z \in \mathcal{P}_{\text{rep}}$,
- there exists $m \in \mathbb{N}$ such that $f^m(z) \in \mathcal{P}_{\text{attr}}$,
- $h(\zeta) = \Phi_{\text{attr}}(f^m(z)) - m + n$.

We have illustrated a possible orbit on Figure 2.

The iterative residue: Let

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

be the power series expansion of f . The iterative residue of f is the quantity $\gamma = 1 - \frac{a_3}{a_2^2}$. It is related to the residue at 0 of the meromorphic form $\frac{dz}{f(z)-z}$ by the following formula: $\frac{1}{2\pi i} \oint \frac{dz}{f(z)-z} = \gamma - 1$. In fact the (multivalued near the origin) primitive $\int \frac{dz}{f(z)-z} + \frac{dz}{z}$ turns out to be an interesting approximation of the Fatou coordinates, as their expansions share the same first two terms: as z tends to 0 within a closed sector avoiding the repelling axis for $\Phi = \Phi_{\text{attr}}$ or the attracting axis for $\Phi = \Phi_{\text{rep}}$:

$$\Phi(z) = \frac{-1}{a_2 z} - \gamma \log z + \text{constant} + o(1).$$

Another characterization is in terms of the horn map: there are expansions

$$\begin{aligned} h(z) &= z + a_{\text{up}} + o(1) & \text{as } \text{Im}(z) \rightarrow +\infty \\ h(z) &= z + a_{\text{down}} + o(1) & \text{as } \text{Im}(z) \rightarrow -\infty \end{aligned}$$

The constants a_{up} and a_{down} depend on the normalization of Fatou coordinates, but not the quantity $a_{\text{up}} - a_{\text{down}}$. It turns out that

$$a_{\text{up}} - a_{\text{down}} = -2\pi i \gamma.$$

Interestingly, if we consider the horn map with the normalization number 2 presented below, then $a_{\text{up}} = -\pi i \gamma$ and $a_{\text{down}} = \pi i \gamma$.

Some normalizations: We will give here three examples of normalizations for the upper parabolic renormalization of f . The first two work well for germs,⁴ the third makes strong structural assumptions on f . Let $\mathcal{R}[f]$ temporarily denote any upper parabolic renormalization (i.e. without normalization). So this map is defined up to pre and post composition by two linear maps. Let

$$\begin{aligned} f(z) &= z + a_2 z^2 + a_3 z^3 + \dots \\ \mathcal{R}[f](z) &= b_1 z + b_2 z^2 + \dots \end{aligned}$$

be their power series expansions. Here are our examples of normalizations:

- (1) By imposing $b_1 = 1$ and $b_2 = 1$: this first approach is easier but assumes that $b_2 \neq 0$. Then there is a unique pair of linear maps A, B such that $A \circ \mathcal{R}[f] \circ B(z) = z + z^2 + \mathcal{O}(z^3)$.
- (2) By normalizing the expansion of the Fatou coordinates: Fatou coordinates are unique up to addition of a constant. Moreover, the following limited expansion is valid (even though there is not a *convergent* power series expansion in general): on all closed sectors avoiding respectively the repelling and the attracting axis, we have, as $z \rightarrow 0$:

$$\begin{aligned} \Phi_{\text{attr}}(z) &= \frac{-1}{a_2 z} - \gamma \log_p \frac{-1}{a_2 z} + \text{constant} + o(1) \\ \Phi_{\text{rep}}(z) &= \frac{-1}{a_2 z} - \gamma \log_p \frac{1}{a_2 z} + \text{constant} + o(1) \end{aligned}$$

where \log_p denotes the principal branch of the logarithm. The normalization just consists in adding constants to both Fatou coordinates so as to cancel the two constants in the above expansions. This normalizes $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$ and we then choose $\mathcal{R}[f]$ to be the semi-conjugate of h by $E : z \mapsto e^{2\pi i z}$. Note that with this normalization,

$$\begin{aligned} h(z) &= z - i\pi\gamma + o(1) \text{ as } \text{Im } z \rightarrow +\infty \text{ and} \\ h(z) &= z + i\pi\gamma + o(1) \text{ as } \text{Im } z \rightarrow -\infty. \end{aligned}$$

⁴We use the word *germ* in the following meaning: an equivalence class of holomorphic maps defined near the origin, with $f \sim g$ if they coincide in some neighborhood of 0. This is equivalent to f and g having the same power series expansion at the origin.

where γ is the iterative residue.

- (3) By the critical value: we will meet later in this article a class of maps whose renormalizations have a unique critical value. The normalization $A \circ \mathcal{R}[f] \circ B$ then chooses the linear maps A so as to place the critical value at 1 and then B so that $A \circ \mathcal{R}[f] \circ B$ has derivative 1 at the origin.

Each of these conventions has its own advantages. Let \mathcal{R}_{nor} denote in this paragraph the normalized upper renormalization of f . Conventions number 1 and 3 have the property that $\mathcal{R}_{\text{nor}}[g \circ f \circ g^{-1}] = \mathcal{R}_{\text{nor}}[f]$ in a neighborhood of 0 for all holomorphic maps g fixing the origin with $g'(0) \neq 0$. They also give back a parabolic germ $\mathcal{R}_{\text{nor}}[f]$. Number 2 does not necessarily, but it is defined for all f . We will work with a class of maps satisfying number 3. Our choice in most of the article will be to normalize Fatou coordinates and the horn map according to number 2, and the parabolic renormalization according to number 3.

1.3. What are horn maps good for? Horn maps occur in at least two ways:

- First as local conjugacy invariants. A complete local conjugacy invariant of a parabolic germ with one petal is more or less given by the data of the pair of germs of its horn maps at both ends of the cylinder (see [Vor81] for precise statements; [MR83] gives an interesting equivalent point of view).
- Second as limits of cylinder renormalization. If a sequence of maps f_n tends to f and fix the origin with multiplier λ_n and if $2\pi i/(\lambda_n - 1) = N_n + a + o(1)$ with $N_n \in \mathbb{Z}$, $N_n \rightarrow \pm\infty$ and $a \in \mathbb{C}$, under some mild supplementary assumptions, the fixed point of f at the origin is the limit of a pair of fixed points of f_n , the origin and another one, and is possible to draw crescent shaped domains with tips at these two fixed points delimited by a curve C_n and its image $f_n(C_n)$. The quotient of this domain by identifying $z \in C_n$ with $f_n(z)$ is isomorphic as a Riemann surface to the cylinder \mathbb{C}/\mathbb{Z} . The first return map from the cylinder to itself then tends, as $n \rightarrow +\infty$, to the horn map (up to pre and post composition with translations). See [Dou94, Shi00, IS04].

The second point justifies why it makes sense to iterate horn maps.

A very important application comes from Lavaurs' theorem: let $\sigma \in \mathbb{C}$ and let the Lavaurs map g_σ be defined as

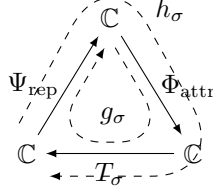
$$g_\sigma = \Psi_{\text{rep}} \circ T_\sigma \circ \Phi_{\text{attr}}$$

where $T_\sigma(z) = z + \sigma$. Then under the same assumptions as above, $f_n^{N_n} \rightarrow g_\sigma$ for some value of σ that depends on a (and on the chosen normalizations of the Fatou coordinates). This is why the Lavaurs maps are also called *geometric limits* by analogy with the field of Kleinian groups. Application of Lavaurs's theorem include parabolic enrichments (understanding the Hausdorff limits of Julia sets of a sequence of polynomials tending to one with a parabolic point), non local connectedness of some bifurcation loci, and several discontinuity theorems.

Now horn maps are closely related to Lavaurs maps because each are semi-conjugate of the other. More precisely, consider the following non-commuting diagram:

$$\begin{array}{ccc} & \mathbb{C} & \\ \Psi_{\text{rep}} \nearrow & & \searrow \Phi_{\text{attr}} \\ \mathbb{C} & \xleftarrow{T_\sigma} & \mathbb{C} \end{array}$$

The map g_σ is the composition obtained by starting from the top node, and following the arrows in a loop back to the starting node. The map $h_\sigma := T_\sigma \circ h$ is the same but starting from the lower left corner.



Following one resp. two arrows from one corner to another gives a semi-conjugacy from h_σ to g_σ resp. from g_σ to h_σ . The first advantage of horn maps over Lavaurs maps is that they are easier to understand and have better covering properties in many applications (the best is to project the extended horn maps, they commute with T_1 , down to a dynamical system on \mathbb{C}/\mathbb{Z}). From this stems a second advantage: the invariance under parabolic renormalization of some classes of maps, as explained in the following sections.

1.4. A reminder about singular values of maps. Let $f : X \rightarrow Y$ be a holomorphic map where X and Y are Riemann surfaces. Let us recall that a *singular value* of f , as a map from X to Y , is an element $z \in Y$ which has no open neighborhood over which f is a cover⁵. Every critical value is singular, as is every asymptotic value⁶, and it is a simple yet very useful theorem that the set of singular values is the closure of the set of all critical and asymptotic values.

It shall be noted that restricting the domain of a map will likely introduce a lot of singular values: if $U \subset X$, every point in $f(\partial U)$ will be a singular value of f as a map from U to Y . Similarly, enlarging the range Y will introduce singular values at boundary points.

1.5. Universality and maps with all “the” structure. For $d \geq 2$ an integer, let

$$B_d(z) = \left(\frac{z+a}{1+az} \right)^d \quad \text{with } a = a_d = \frac{d-1}{d+1}.$$

Let

$$B_\infty(z) = \exp \left(2 \frac{z-1}{z+1} \right).$$

They induce unisingular self maps of \mathbb{D} with a unique singular value $z = 0$ in \mathbb{D} and they have a parabolic fixed point on the boundary at $z = 1$ with two attracting petals. Interestingly:

$$B_d \xrightarrow{d \rightarrow +\infty} B_\infty$$

uniformly on compact subsets of \mathbb{D} .

The unit disk is the (immediate) basin of one petal. The inverse of the unit disk is the basin of the other. We let $\Phi_{\text{attr}}[B_d] : \mathbb{D} \rightarrow \mathbb{C}$ be the extended attracting Fatou coordinate for the first petal. The map has also two repelling petals, with vertical axes. We choose the one on the top and let $\Psi_{\text{rep}}[B_d]$ denote the corresponding extended repelling Fatou coordinate. We let $h[B_d] = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. It is defined on an upper half plane.

Theorem 1 (folk). *Let $f : U \subset \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ a holomorphic map with a parabolic petal of period one and such that one and only one singular value of f , as a map from U to $\widehat{\mathbb{C}}$, lies in the associated immediate basin A . Then the restriction of f to A is analytically conjugated to the restriction of B_d to \mathbb{D} for some $d \in \{2, 3, \dots\} \cup \{\infty\}$.*

⁵i.e. there is no open subset V of Y containing z s.t. f is a cover from $f^{-1}(V)$ to V . The definition is equivalent if we consider only neighborhoods V of z homeomorphic to disks

⁶ $z \in Y$ is an asymptotic value whenever there exists a continuous path $\gamma : [0, t[\rightarrow X$ that leaves every compact of X and whose image by f tends to z

See for instance [DH85], exposé IX for a similar statement. This has the following consequences, discovered by several authors, including Shishikura (see [Shi94]), and Lanford and Yampolsky (see [LY14]). See also Part 3 of [Ché01]

Corollary 3 (S., L.-Y.). *With the same notations, call $\zeta : A \rightarrow \mathbb{D}$ the conjugacy from f to B_d . Then there exists a constant $\tau \in \mathbb{C}$ (which depends on the normalizations of the Fatou coordinates) such that $\Phi_{\text{attr}}[B_d] \circ \zeta = \tau + \Phi_{\text{attr}}[f]$, where the right hand side is restricted to A .*

Thus in particular, using the terminology introduced here, $\tau + \Phi_{\text{attr}}[f]$ restricted to A is structurally equivalent to $\Phi_{\text{attr}}[B_d]$ over \mathbb{C} . This is illustrated on Figures 4, 5, 6 and 7, using a widespread visualization technique explained in Section 2.

Theorem 2 (S., L.-Y.). *For a fixed d , all the maps in the situation of Theorem 1 and such that the concerned parabolic point has only one attracting petal⁷ have structurally equivalent upper renormalizations, when the latter is normalized by setting the singular values to 0, 1 and ∞ . More precisely they are (I, Y) -structurally equivalent with $Y = \mathbb{C}$, I being a singleton and the marked point being the origin. The same holds for the lower renormalization, and the upper one is structurally equivalent to the conjugate of the lower by the reflection $z \mapsto 1/\bar{z}$. Moreover the upper or lower parabolic renormalization $g : V \rightarrow \widehat{\mathbb{C}}$ is defined on a simply connected set and has exactly 3 singular values: the asymptotic values 0, ∞ and one critical value if d is finite or one singular value otherwise.*

This is illustrated in Figures 16 and 17.

For later reference, let us mention the following

Complement (S., L.-Y.). *Recall $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. Let f be a holomorphic map as in Theorem 2. Let $U[f]$ denote the component of the domain of $h[f]$ that contains an upper (resp. a lower) half plane. (Up to a complex rescaling, resp. an inversion and a complex rescaling, the image of $U[f]$ by $E : z \mapsto e^{2\pi iz}$ is the domain of the renormalization of f .) Then there is a conformal isomorphism $\phi_0 : U[f] \rightarrow U[B_d]$ that commutes with T_1 and such that $\Psi_{\text{rep}}[B_d] \circ \phi_0 = \zeta \circ \Psi_{\text{rep}}[f]$, where $\zeta : A[f] \rightarrow A[B_d]$ is the conjugacy of the immediate parabolic basins of the respective fixed petals, mentioned in Theorem 1.*

We will use Theorem 2 in conjunction with

Theorem 3 (Fatou+folk). *Let $f : U \subset \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ a holomorphic map with a parabolic fixed point. Then either at least one singular value of f belongs to each cycle of immediate parabolic basins, or $U = \widehat{\mathbb{C}}$ and f is a homography.*

Consequence: the structural equivalence classes mentioned in Theorem 2 are stable by parabolic renormalization. For what we are concerned with in this article, this is the base of everything.

1.6. Inou and Shishikura: giving up part of the structure to gain flexibility. Here is the central gear in the work of Inou and Shishikura:

Theorem 4 (Inou Shishikura). *There exists an explicit pair of open subsets A, B of \mathbb{C} and an explicit holomorphic map $f_0 : B \rightarrow \mathbb{C}$ with the following properties:*

- (1) $0 \in A$, A is compactly contained in B ,
- (2) A and B are simply connected,
- (3) f_0 fixes the origin and has derivative 1 there,

⁷This condition is not necessary: we added it because we defined parabolic renormalization in the present article only for parabolic points with one attracting petal. See Figure 11 for a case where the parabolic point has three attracting petals.

- (4) f_0 has exactly one critical point in B ; it has local degree two and it belongs to A , and is mapped to $-4/27$ by f_0 ,
- (5) for any upper renormalization g of a map satisfying the hypotheses of Theorem 2, there exists a subset U of $\text{Dom } g$ and a holomorphic bijection $\phi : B \rightarrow U$ with $\phi(0) = 0$ and $g|_U = f_0 \circ \phi^{-1}$
- (6) for any univalent map $\phi : A \rightarrow \mathbb{C}$ with $\phi(0) = 0$ and $\phi'(0) = 1$, there exists a univalent map $\psi : B \rightarrow \mathbb{C}$ with $\psi(0) = 0$ and $\psi'(0) = 1$, such that the map $f_0 \circ \phi^{-1}$, which fixes the origin with multiplier one, has an upper renormalization which has a restriction of the form $f_0 \circ \psi^{-1}$.

The map f_0 has a particularly simple expression: $f_0(z) = z(1+z)^2$. It turns out that f_0 commutes with $z \mapsto \bar{z}$, thus the theorem holds with the same f_0 for lower renormalization.

The statement below is a reformulation of their theorem using the language introduced in the present article. Given a structure \mathcal{B} and a sub-structure \mathcal{A} , we will say that the second is a *relatively compact sub-structure*⁸ of the first whenever maps in \mathcal{A} are structurally equivalent to restrictions of maps in \mathcal{B} to relatively compact open subsets of their domains (not just subsets).⁹

Theorem 5 (Inou Shishikura, reformulated). *Let I be a singleton and $Y = \mathbb{C}$. There exists an explicit pair of (I, Y) -structures \mathcal{A} and \mathcal{B} with the following properties:*

- (1) \mathcal{A} is a relatively compact sub-structure of \mathcal{B} and \mathcal{B} is a sub-structure of the universal structure of Theorem 2,
- (2) $\forall (a, f) \in \mathcal{A}$, the map f is defined on a connected and simply connected Riemann surface and has exactly one critical point, of local degree two; the same holds for \mathcal{B} .
- (3) For any map in \mathcal{A} whose domain of definition is a subset of \mathbb{C} and that fixes the marked point with multiplier one, its (suitably normalized) parabolic renormalization has at least structure \mathcal{B} .

Definition 4 (High type numbers). *For $N \in \mathbb{N}^*$, let HT_N be the set of irrationals whose modified continued fraction satisfies $|a_n| \geq N$, $\forall n \in \mathbb{N}$. In settings where N has been fixed, the set HT_N is often called the set of high type numbers. We will call it here the set of numbers of type $\geq N$.*

To keep it short, the following corollary, also by Inou and Shishikura, is stated here with some imprecision concerning the renormalization:

Corollary 5 (Perturbation). *There exists $N > 0$ such that the class of maps defined in an open subset of \mathbb{C} , with structure \mathcal{A} and fixing the marked point with a rotation number θ of type $\geq N$, is invariant under a cylinder renormalization operator (called the near-parabolic renormalization).*

They prove more: thanks to the compact inclusion of structure \mathcal{A} in \mathcal{B} , there is a form of contraction. Cylinder renormalization was introduced by Yampolsky in the study of analytic circle homeomorphisms with a critical point.

Consequences of this corollary are numerous and are still being harvested. Its main quality is that it allows a fine control on the post-critical set of quadratic polynomials with high type rotation numbers. For instance, Shishikura proved that in this case the boundary of the Siegel disk is a Jordan curve (unpublished). It allows to study the hedgehogs and the size of Siegel disks. In a recent preprint,

⁸The author does not particularly like the terminology he just introduced.

⁹If it holds for some representatives then it holds for all representatives in the equivalence class.

[CC15] proved the Marmi Moussa Yoccoz conjecture restricted to high type numbers. Cheraghi has given many other applications [Che10, Che13, AC12]. This tool should also allow one to make progresses on the MLC conjecture. It was also used in [BC12] to prove the existence of quadratic polynomials with a Julia set of positive Lebesgue measure. We believe that it can also give a new approach to the results of McMullen [McM98] on the self similarity of Siegel disks whose rotation number has an eventually periodic continued fraction expansion¹⁰ at the critical point. McMullen used Ghys' quasiconformal surgery procedure as a first step in his proofs, to transfer some properties that are easier to prove for circle maps. It would be nice to have a more direct proof, that would adapt to situation, like the exponential maps $z \mapsto e^z + c$, where a quasiconformal surgery does not exist but where self similarity still seems to occur.

1.7. Main Theorem. In this article, we prove the following extension of Inou and Shishikura's Theorem.

Main Theorem. *Let I be a singleton and $Y = \mathbb{C}$. For all $d \in \mathbb{N}$ with $d \geq 2$, there exists (not completely explicit) (I, Y) -structures $\mathcal{A} = (a, f)$ and \mathcal{B} with the following properties:*

- (1) \mathcal{A} is a relatively compact sub-structure of \mathcal{B} and \mathcal{B} is a sub-structure of the universal structure of Theorem 2,
- (2) every map in \mathcal{A} or in \mathcal{B} is defined on a connected and simply connected Riemann surface,
- (3) every map in \mathcal{A} or in \mathcal{B} has exactly one critical value, and all critical points have local degree d ,
- (4) for any map in \mathcal{A} whose domain of definition is a subset of \mathbb{C} and that fixes the marked point with multiplier one, the upper parabolic renormalization has a at least structure \mathcal{B} , and the lower parabolic renormalization has at least structure the conjugate of \mathcal{B} by $z \mapsto \bar{z}$, for appropriate normalizations of the renormalizations.

The structures \mathcal{A} and \mathcal{B} are obtained by retaining most of the universal structure (call it \mathcal{U}) of Theorem 2. More precisely we choose for \mathcal{B} the restriction of a map in \mathcal{U} to a subset of its domain U defined as points having U -hyperbolic distance $\leq L$ to the marked point, and we prove in Section 3 that for L big enough, there is a compact sub-structure \mathcal{A} of \mathcal{B} such that the main theorem holds.

Remark. It should be noted that for $d = 2$, our theorem can be considered as weaker than Inou and Shishikura's. For one thing, maps in our class have much more structure, so our class is smaller. Second they have many critical points (though only one critical value), whereas there is only one in Inou and Shishikura's. This should not prevent our class, though, to be applied to $z^d + c$ as we explain now. Note that a similar situation occurs for the IS class: a polynomial $z^2 + c$ with an indifferent fixed point of multiplier close to 1 never has a restriction that belongs to the IS class, but its first cylinder renormalization has some as soon as the multiplier is close to 0. Here it is the same: a map of the form $z^d + c$ never has structure \mathcal{A} or more, but its first cylinder renormalization does if the rotation number is close enough to 0.

It should be easy to check that the analog of Corollary 5 also holds. We believe that many of its consequences for quadratic maps therefore carry over to unicritical polynomials.

About unisingular maps:

¹⁰these rotation numbers are the quadratic irrationals

We have good hopes to extend the above work to the case $d = +\infty$. There are some subtleties occurring here.

We do not believe that one can take a substructure of $f \in \mathcal{F}$ defined by a restriction on a compact subset of the domain of f , like we did in the case $d < +\infty$. The natural idea is to keep a whole connected preimage of a neighborhood of the singular value, which adds a subset of $\text{Dom } f$ that is at least as tangent to its boundary as a horocycle. Unfortunately, we realized that this does not provide an invariant class. However, we have hopes to find an appropriate sub-structure.

It shall be noted that some consequences of Inou and Shishikura's invariant class for $d < +\infty$ won't hold anymore for $d = +\infty$: for instance there are unisingular maps for which the boundary of the Siegel disk is not a Jordan curve. This includes the exponential $z \mapsto \lambda(\exp(z) - 1)$ (or equivalently $z \mapsto e^z + \kappa$) when it has an indifferent periodic point of rotation number in Herman's class¹¹. Interestingly, there are some other maps with only one active singular value, with $d = +\infty$, and for which the Siegel disk seems to be more often locally connected (always): for instance the semi-conjugate of $z \mapsto e^{i\theta/2} \tan z$ by $z \mapsto z^2$, i.e. $z \mapsto e^{2\pi i\theta} (\tan \sqrt{z})^2$.

It is to be noted that though the two (essentially) unisingular families $\lambda(e^z - 1)$ and $\lambda(\tan \sqrt{z})^2$ have very different Siegel disks for $\theta =$ the golden mean, computer experiments weakly hint at a possible identical asymptotic limit when zooming at the singular value: there might exist a cylinder renormalization operator with a fixed point capturing both maps.

¹¹By [Her85] the Siegel disk is unbounded and then by [BW91] it is not locally connected. Thanks to Lasse Rempe for pointing this out to me.

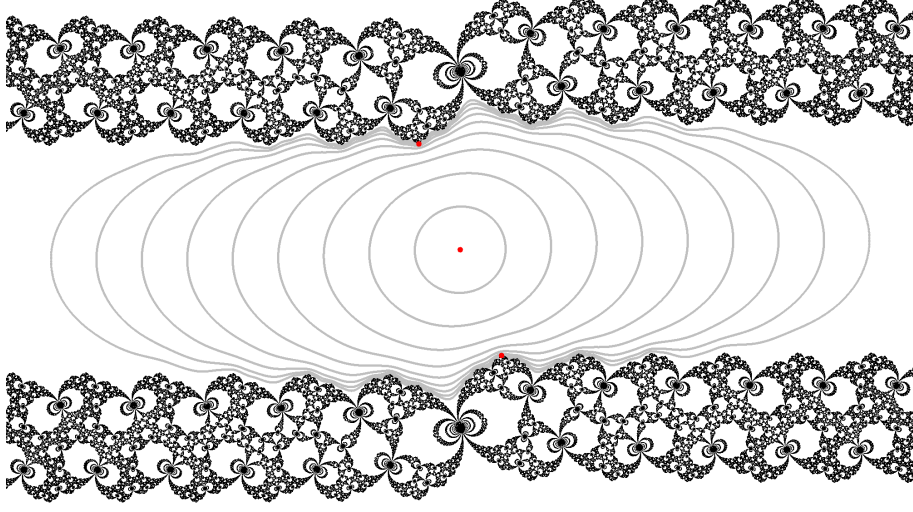


Figure 3: Rotated by 90° , the Julia set of the map $z \mapsto \lambda \tan z$ with λ so that the origin is indifferent with rotation number $\theta/2$ and $\theta = (\sqrt{5} - 1)/2$ is the golden mean. The Julia set is periodic of period π , we drew only two periods. There also seems to be an asymptotic similarity by some imaginary translation. There are red points at the origin and at the two (symmetric) asymptotic values. A few orbits inside the Siegel disk have been drawn. The Siegel disk seems to be bounded by a Jordan curve (but not a quasicircle: there must be a dense set of cusps). The rotation number is $\theta/2$ but the picture has a symmetry of order 2 and quotienting out, i.e. semi-conjugating by $z \mapsto z^2$, gives a transcendental meromorphic map $z \mapsto \lambda^2 (\tan \sqrt{z})^2$ with rotation number θ at 0, with infinitely many critical points but that all map to 0, and with only one asymptotic value $-\lambda^2$.

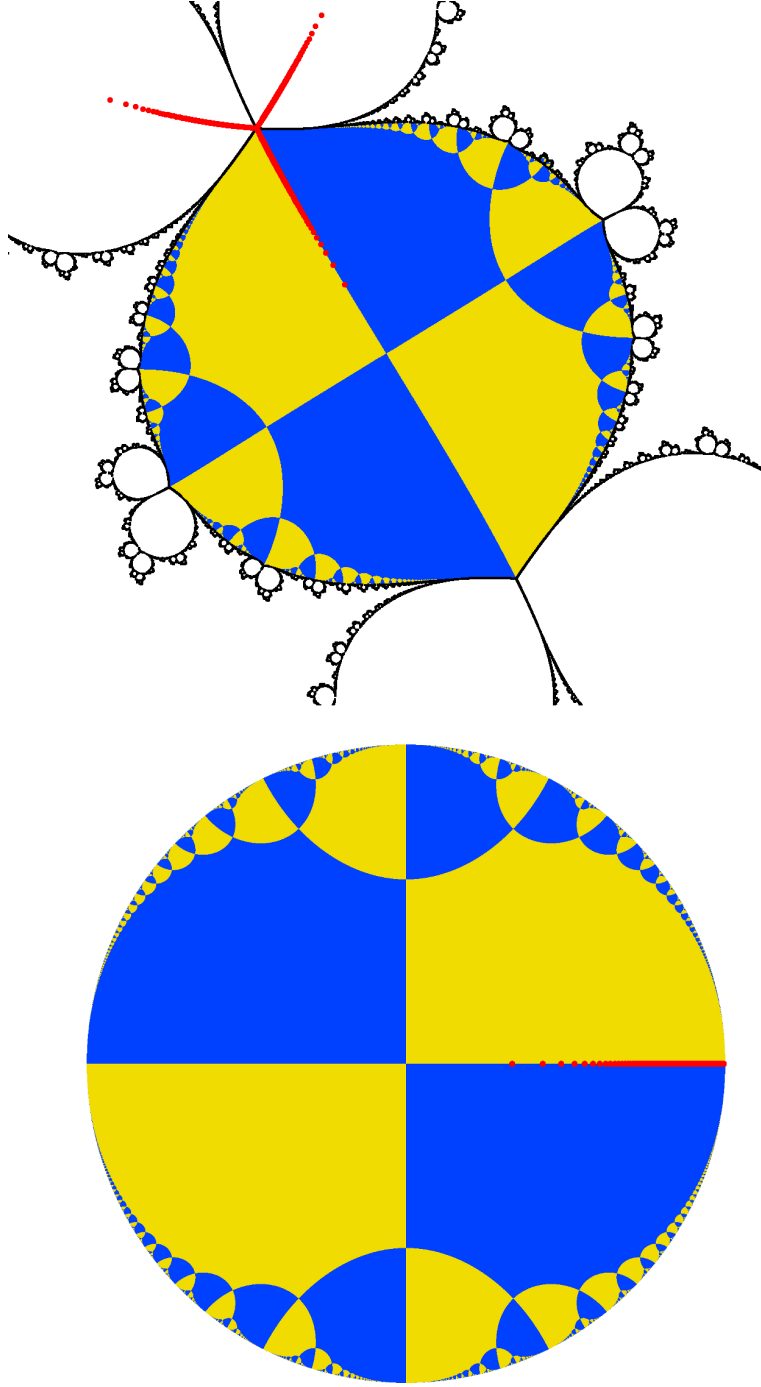


Figure 4: Illustration of Theorem 1. Above: zoom on Douady's fat rabbit, the Julia set of the quadratic map $P = e^{2i\pi/3}z + z^2$, which has a parabolic fixed point with three attracting petals, and acts transitively on them. The Fatou component U that contains the finite critical point has been colored with the parabolic chessboard, whose definition is recalled later in the present article. In red, the orbit of the critical value. On U , P^3 satisfies the hypotheses of the theorem. According to the conclusion, P^3 is conjugated on U to the Blaschke product $B_2(z) = \left(\frac{z+1/3}{1+z/3}\right)^2$. Below: the chessboard of $\mu_2 \circ B_2 \circ \mu_2^{-1}$, with $\mu(z) = (z + 1/3)/(1 + z/3)$, which is conjugated to B_2 and hence to P^3 on U . The conjugacy has to transport the chessboard and the critical orbit.

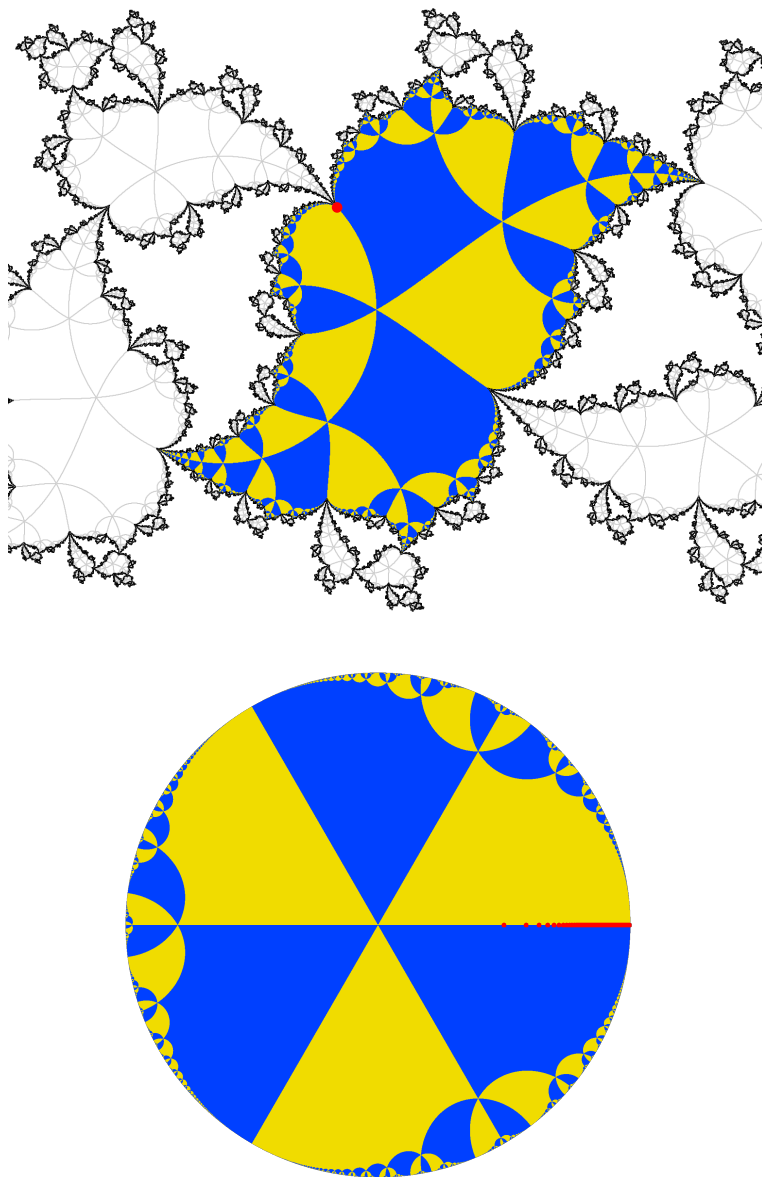


Figure 5: Another illustration of Theorem 1. This time, $d = 3$. The parabolic point on the first picture is indicated by a red dot. The orbit of the critical value is indicated in red on the second picture.

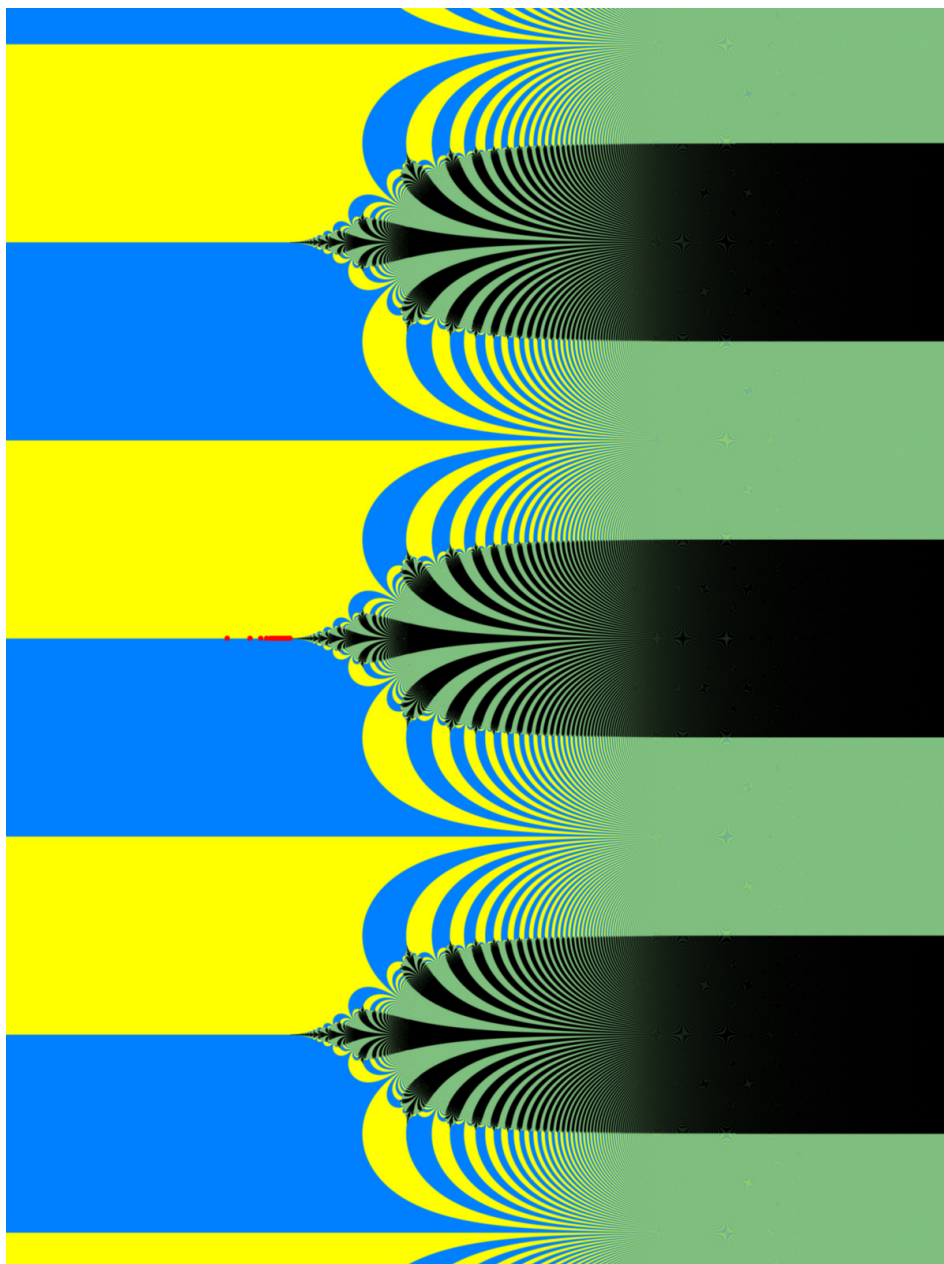


Figure 6: Illustration of Theorem 1 with, $d = \infty$. Caption on Figure 7.



Figure 7: Continuation of Figure 6, for which the map is $z \mapsto e^z - 1$, which has a parabolic fixed point at the origin, with one attracting petal. Its basin is connected. The Julia set is a Cantor bouquet indicated in black (see [Dev99]). The orbit of the singular value is drawn with red dots. The parabolic basin is painted with blue and yellow according to the chessboard partition. The green hues correspond to parts where the yellow and blue are mixed below a pixel's width. For the Julia set, the darker shades correspond to places where the Julia set, thickened by an amount comparable to a fraction of a pixel's size, gets denser. On the present figure, we drew the parabolic chessboard of B_∞ and the orbit of its singular value 0. The conjugacy maps the chessboards and singular orbits of each map to that of the other. Looking at the pictures, the correspondence is not so obvious at first sight.

2. VISUALIZING STRUCTURES

An often used and very useful technique of visualization of ramified covers (and partial cover structures that are not too messy) consists in cutting the range in domains, often simply connected, along lines joining singular values, and taking the pre-image of these pieces, which gives a new set of pieces. The way they connect together and the way they map to the range gives information about the structure.

2.1. Changes of variables. The map B_d is the composition of the automorphism $\mu : z \mapsto \frac{z+a}{1+az}$ of the disk, with $a = (d-1)/(d+1)$, followed by $\text{pow} : z \mapsto z^d$. $B_d = \text{pow} \circ \mu$. If we conjugate B_d by μ we get the map $\mu \circ \text{pow}$:

$$\tilde{B}_d = \mu \circ B_d \circ \mu^{-1} : z \mapsto \frac{z^d + a}{1 + az^d}$$

which is a Blaschke product too and has its critical point at the origin, the parabolic point still being at $z = 1$. As $d \rightarrow +\infty$, \tilde{B}_d tends to the constant 1 uniformly on compact subsets of \mathbb{D} .

The map B_∞ is the composition of $\mu : z \mapsto i\frac{1-z}{1+z}$ (mapping the disk to the upper half plane) followed by $z \mapsto \exp(2iz)$. Interestingly, if we conjugate B_∞ with μ we get the trigonometric map:

$$\mu \circ B_\infty \circ \mu^{-1} : z \mapsto \tan z$$

whose parabolic fixed point is at the origin and which maps the upper half plane to itself, the singular value in the upper half plane being i .

2.2. Preferred representative. Theorem 2 says that all maps satisfying some assumption have structurally equivalent upper parabolic renormalization. Their equivalence class, that depends only on d , has something universal. We will here choose a preferred representative, and for this use the maps B_d . A defect of the maps B_d and \tilde{B}_d , seen as maps from the Riemann sphere to itself, is that their parabolic point has two attracting petals instead of one. We prefer to work with a semi-conjugate of B_d that we introduce now. The map B_d commutes with $z \mapsto 1/\bar{z}$ and with $z \mapsto \bar{z}$ hence with $z \mapsto 1/z$. It is therefore a well defined map on pairs $\{z, 1/z\}$. A first change of variables $u = (1-z)/(1+z)$ maps the unit disk to the right half plane “ $\text{Re}(z) > 0$ ” sending the parabolic point to 0 and conjugates B_d to a map which can be formulated as follows:

$$u \mapsto \frac{\text{odd} \left(\left(1 + \frac{u}{d}\right)^d \right)}{\text{even} \left(\left(1 + \frac{u}{d}\right)^d \right)}.$$

where odd and even refer to the sum of monomials of odd and even power in u in the polynomial expansion of $(1 + u/d)^d$. For $d = \infty$ we get the ratio of the odd and even parts of the exponential, a.k.a.

$$u \mapsto \tanh(u).$$

Setting $v = -u^2$ identifies pairs $\{z, 1/z\}$ with single values of v . There exists a map C_d , rational of degree d if $d < \infty$, entire transcendental if $d = \infty$, such that the following diagram commutes

$$\begin{array}{ccc} & B_d & \\ S \downarrow & \square & \downarrow S \\ & C_d & \end{array}$$

where $S(z) = v = (i(1-z)/(1+z))^2$. If $d = \infty$ we get $C_\infty(v) = (\tan \sqrt{v})^2$. If $d < \infty$ the formula is more complicated. The map S is a bijection from the unit disk to the

complement A of $[0, +\infty]$ in the Riemann sphere, and sends 1 to 0, -1 to ∞ and 0 to -1 . The map C_d has a parabolic fixed point at the origin with one attracting petal, whose immediate basin is A . By construction C_d is conjugate on the basin to the restriction of B_d to \mathbb{D} . The extended horn map of C_d is defined on the complement of a horizontal line. Thus the upper and lower parabolic renormalizations of C_d are defined on round disks centered on the origin. A lengthy computation shows that

$$\gamma[C_d] = \frac{3}{20} \cdot \frac{d^2 + 1}{d^2 - 1}.$$

We have defined in Section 1.5 the objects $\Phi_{\text{attr}}[B_d]$, $\Psi_{\text{rep}}[B_d]$ and $h[B_d]$. They are related to C_d as follows:

$$\begin{aligned} \Phi_{\text{attr}}[B_d] &= \Phi_{\text{attr}}[C_d] \circ S|_{\mathbb{D}} \\ S \circ \Psi_{\text{rep}}[B_d] &= \Psi_{\text{rep}}[C_d] \\ h[B_d] &= h[C_d]|_H \end{aligned}$$

where H is the upper half plane on which $h[B_d]$ is defined, and S is the 2:1 rational map defined a few lines above, that semi conjugates B_d to C_d . If we choose a normalization for the objects associated to C_d this induces a normalization for the objects associated to B_d .

2.3. Visualizations. In the case of parabolic renormalizations of maps satisfying the hypotheses of Theorem 2, our preferred visualization works on the cylinder coordinates \mathbb{C}/\mathbb{Z} just before the conjugacy by $E : \mathbb{C}/\mathbb{Z} \rightarrow \mathbb{C}$, $z \mapsto e^{2\pi iz}$ and completion at 0 and ∞ . In fact, we will first look at a visualization before the projection from \mathbb{C} to \mathbb{C}/\mathbb{Z} , i.e. we will look at a visualization of the horn map $h = \Phi \circ \Psi$, where Φ is a shorthand for Φ_{attr} and Ψ for Ψ_{rep} (extended Fatou coordinate and parameterizations).

Let v_f denote the unique singular value of f in the immediate parabolic basin A . The set of singular values of h over \mathbb{C} is of the form $v' + \mathbb{Z}$ for $v' = \Phi(v_f)$. Let us cut the range along the horizontal line $v' + \mathbb{R}$ passing through them. To understand the shape of the preimages of this line and of the upper and lower half planes it bounds, it is useful to work first with the map B_d . Recall: h is the horn map associated to a dynamical system f with an immediate parabolic basin A , on which there is a conjugacy $\zeta : A \rightarrow \mathbb{D}$ to the map B_d , and $\Phi[B_d] \circ \zeta = \tau + \Phi[f]$. Thus the preimage $\Phi[f]^{-1}(v' + \mathbb{R})$ is mapped by the isomorphism ζ to a universal shape, that depends only on d . The set $\Phi[f]^{-1}(v' + \mathbb{R})$ is called the *parabolic chessboard graph* of f on A . The connected components of its complement in A are called the *chessboard boxes* (in an actual chessboard they are called squares but here they have infinitely many corners and not just four). The chessboard is the name of this decomposition of A into a graph and boxes. Since the chessboard is universal, it can be well understood by looking only at the maps B_d . Note that these maps have a singular orbit contained in $[0, 1[$ and that they send reals to reals, thus the chessboard graph is also the union of the preimages of $[0, 1[$.

Case 1: $d = \infty$. We obtain Figure 7.

Case 2: d is finite. Instead of showing the graph for B_d , whose critical value is at the origin we prefer to show it for the conjugate map \tilde{B}_d introduced earlier, for which the critical point is at the origin. The result is given for $d = 2$ and $d = 3$ on Figures 4 and 5.

To the convergence of B_d to B_∞ as $d \rightarrow +\infty$, seems to echo a convergence of the chessboard decomposition: see Figure 8.

Each chessboard box is mapped by Φ to the upper or the lower half plane delimited by $v' + \mathbb{R}$. The set of singular values of Φ is precisely $\{v' - 1, v' - 2, v' - 3, \dots\}$.

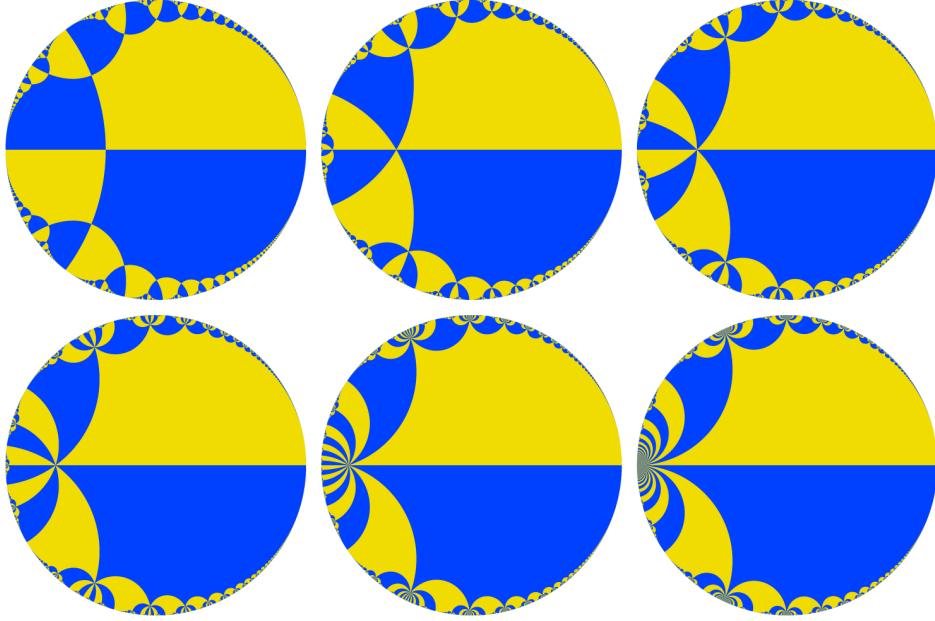


Figure 8: Convergence of the chessboard as $d \rightarrow +\infty$. In reading order, the chessboard of B_d for $d = 2, 3, 4, 5, 10$ and ∞ .

These singular values however have also regular preimages, so these universal structures we are considering are not so simple as ramified covers. Under the dynamics of f , each box is mapped to a box of the same color, and there is exactly one box of each color that is fixed by f : these are the ones that have the singular value in their boundaries. The Fatou coordinate Φ conjugates the dynamics of f on these two fixed boxes to the dynamics of the translation by 1 on the upper and lower half planes. The chessboard also tells us about structure of Φ as defined in Section 1.1. In view of this, the chessboard in the immediate basin A is both a dynamical object w.r.t. f and a structural object w.r.t. Φ .

The figures can be enhanced a little: let us use two shades of yellow and two shades of blue in the range of Φ . Use the light shade if the floor integer part $\lfloor \operatorname{Re}(z - v') \rfloor$ is even, and the dark shade otherwise. Color points in \mathbb{D} according to $\Phi(z)$. Then we get Figure 9. This color scheme is useful to visualize the pull-back by Φ of the vertical direction. Under f , a light strip is mapped to a dark strip and vice-versa.

The chessboard graph has no endpoint, and it is closed in A but not compact. Since we considered the chessboard graph as a subset of \mathbb{C} endowed with its topology, not as a combinatorial object, there is an ambiguity outside branching points concerning which points are vertices of valence 2 and which points belong to edges: the singular value is one such point w . So let us define an abstract graph with vertices at all preimages of the singular value by $\pi \circ \Phi[f] : A \rightarrow \mathbb{C}/\mathbb{Z}$, and edges as preimages of the horizontal circle through it.

Remark. We will not make use of it, but it would make sense to consider some supplementary topological information on the abstract graph, like the cyclic order induced by the embedding in the plane on edges at every vertex.

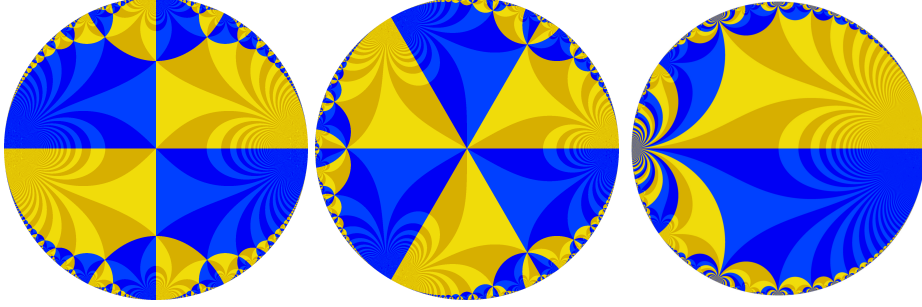


Figure 9: Light and dark strips, preimages of vertical strips of width 1 under the extended attracting Fatou coordinate Φ , \tilde{B}_2 , \tilde{B}_3 and B_∞ .

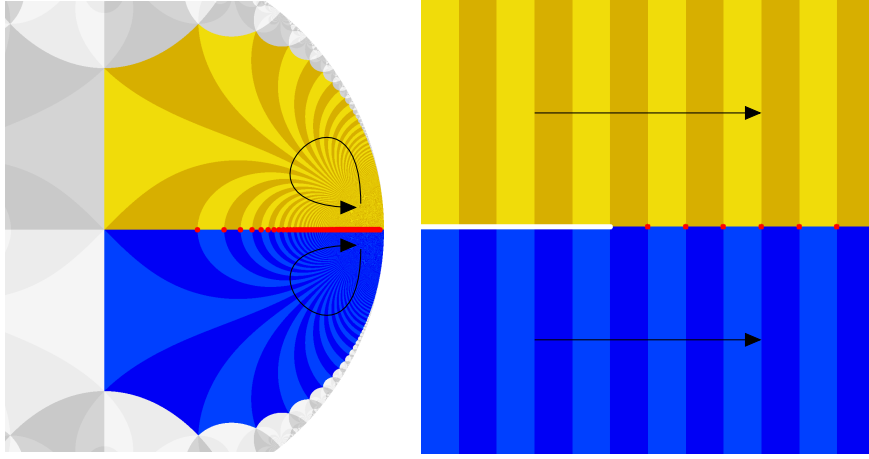


Figure 10: The extended attracting Fatou coordinates of B_2 conjugate the restriction of B_2 to the two principal chessboard boxes, indicated on the left, to the translation $z \mapsto z + 1$ on a slit plane, as on the right.

The graph and the way it is embedded in A also tells us how are glued together pieces obtained by cutting A along the preimages of the vertical line through the critical value of $\pi \circ \Phi[f]$.

Figure 10 explains how the union the edges touching points in the orbit of the singular value form an infinite line in the graph, and how the union of this line and of the two chessboard boxes whose closure contain the line, make a domain where the dynamics is conjugated to the translation by 1 restricted to $\mathbb{C} \setminus]-\infty, 0[$. The bright and dark strips help to figure out how things are mapped and what the dynamics is within this domain. This would work for any $d \geq 2$, including ∞ .

Let us now introduce the chessboard associated to the horn map h . It is defined using the pre-image of the horizontal line through the set of singular values of h , and of the upper and lower half plane cut by this line. From the definitions, it follows that it is also equal to the pre-image by Ψ of the chessboard of f in the full parabolic basin (the union of all iterated preimages of A by f). This time, it is not a dynamically invariant object, but it gives information on the structure of h as defined in Section 1.1.

Figure 11 explains how one can proceed to guess the shape of the domain of the horn map and its parabolic chessboard, using a crescent shaped fundamental domain for the repelling Fatou coordinates. In the picture we only looked at the

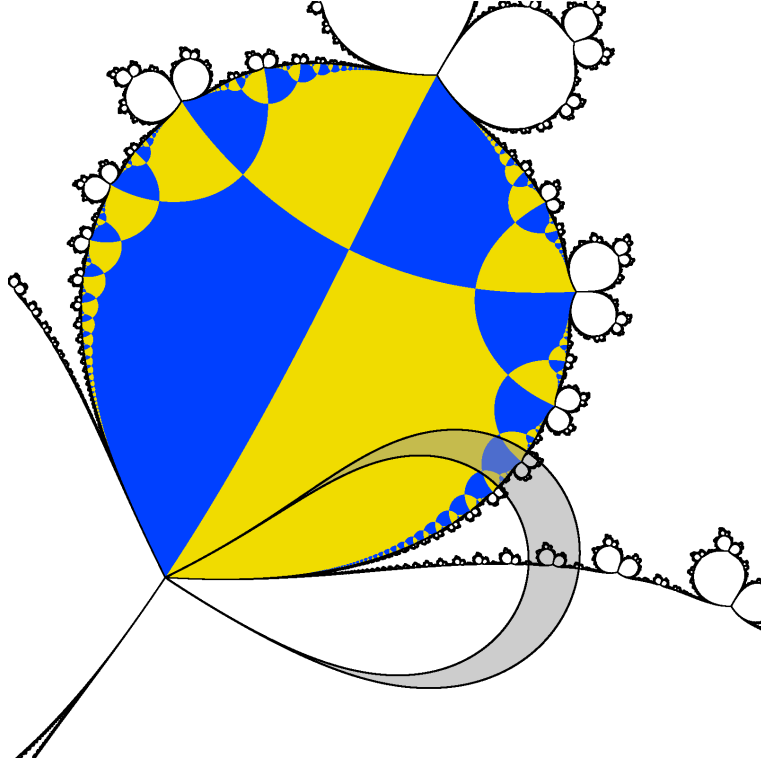


Figure 11: This figure shows a grayed out fundamental domain (accurately computed) for the repelling Fatou coordinates, more precisely an open set mapped conformally by $\Phi_{\text{rep}}[f]$ to a vertical strip of the form $a < \text{Re}(z) < a + 1$ for some $a \in \mathbb{R}$. The map f is the third iterate of the Douady's Rabbit polynomial and we chose U to be out one of the three immediate basins of the parabolic point, which is near the lower left corner. This kind of drawings help to understand the structure of the extended horn map. See the text page 22.

immediate basin. Note that we only defined the horn map for maps with one attracting petal attached to the parabolic point, whereas Figure 11 shows an example with three. Here, there are several inequivalent definitions for the horn map. Let us give one such that the domain of the horn map is the smallest still giving a parabolic renormalization with the full structure: $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$ where Φ_{attr} is the extended attracting Fatou coordinate restricted to the immediate basin A of the petal, and Ψ_{rep} is the repelling Fatou coordinate associated to one of the two repelling petals adjacent to A .

The next set of pictures, in Figure 12, shows the structure of the horn maps of B_d . The image of these three pictures by the exponential map $z \mapsto \exp(2\pi iz)$ is shown on Figures 13, 14 and 15, and gives us information about the structure of the upper renormalization $\mathcal{R}[B_d]$ of B_d . One sees that it is also defined on a disk centered on the origin. For the beauty of the thing, we replaced the dark and light strips by a lighting scheme that gives the illusion of a texture made of cylinders.¹² A more shameful reason for this change is that the light and dark scheme does not pass to the quotient. Let us explain a bit more these pictures: recall that there are

¹²The trick to produce such a computer picture is called *normal mapping*, it is the same trick used to give a realistic look in 3D imaging to texture-mapped polygons subjected to a light source. Some specular reflection reinforces the feeling of relief.

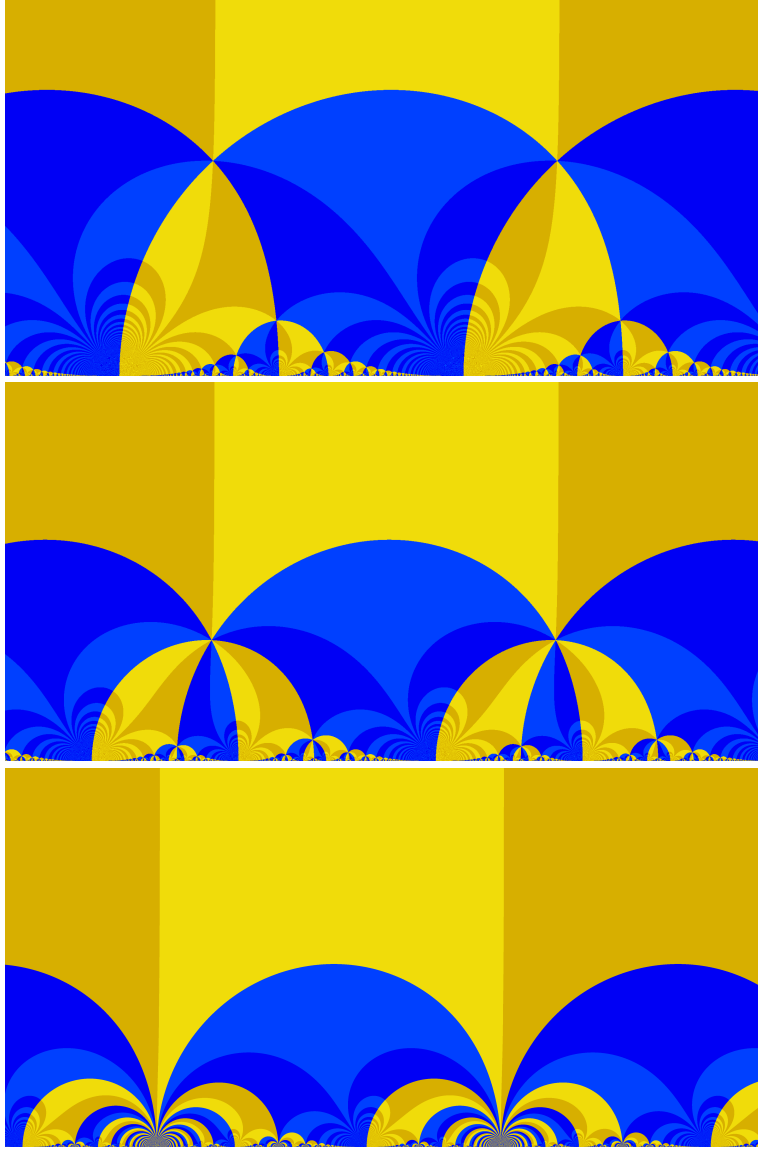


Figure 12: These three pictures show the structures of the extended horn maps h of respectively B_2 , B_3 and B_∞ . They are all defined on the complement of a horizontal line; in each case, we only drew the picture above this line; the full picture is obtained by reflection through this line, permuting blue \leftrightarrow yellow. The same coloring conventions apply as in the previous figures: yellow boxes map by h to the upper half plane delimited by the horizontal line through the singular value, blue boxes to the lower half plane. The boundaries between dark and light shades are mapped by $\pi \circ h$ to the vertical lines through the critical value.

three singular values of $\mathcal{R}[B_d]$: 0 , ∞ and some third point ν . While the horn map goes from repelling to attracting Fatou coordinates, the parabolic renormalization goes from a space to itself, so it makes sense to compare the position of the singular value and the radius r of the disk where B_d is defined. It turns out that, because of the requirement that the renormalization has a parabolic fixed point at the origin,

Figure 13: Structure of $\mathcal{R}[B_2]$.

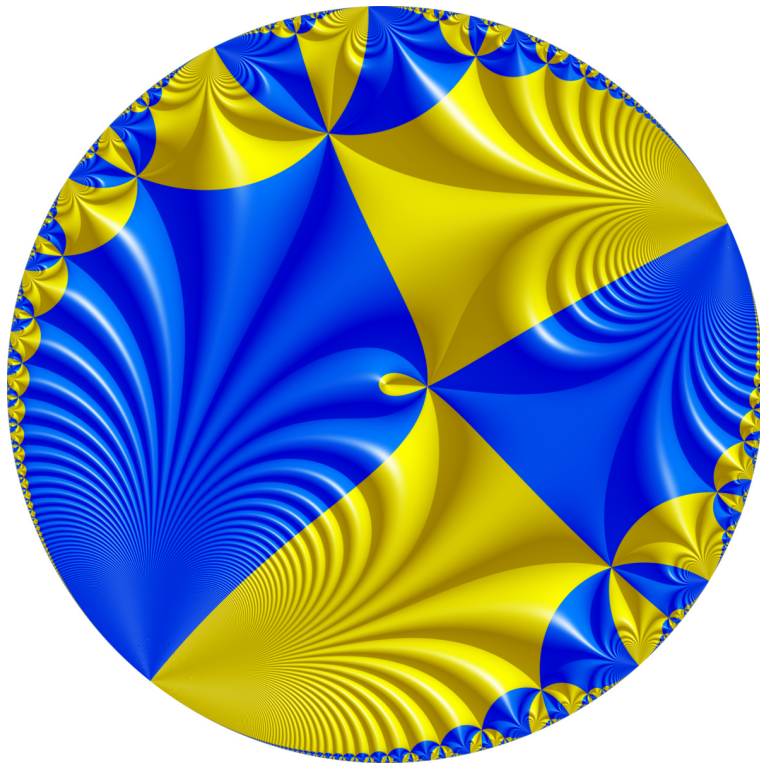
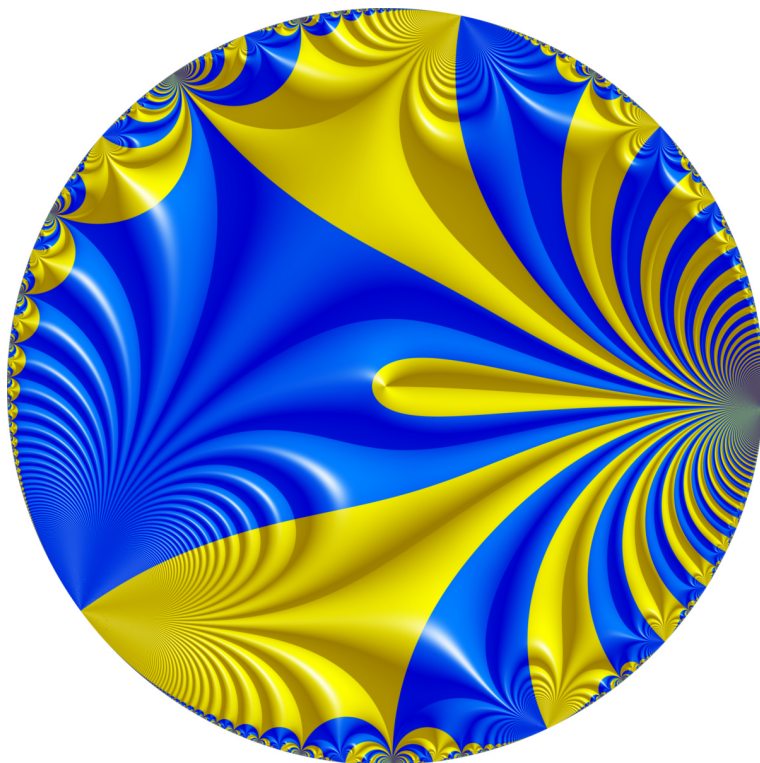
$|\nu|$ is notably smaller than r , as was already remarked by several people before:

$$|\nu|/r = e^{-2\pi^2 \operatorname{Re}(\gamma)}$$

where γ denotes the iterative residue of C_d (see Sections 1.2 and 2.2). Let us quickly justify this: the horn map $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$ has expansion $h(z) = z + a_{\text{up}} + o(1)$ when $\operatorname{Im}(z) \rightarrow +\infty$ and if Φ_{attr} and Ψ_{rep} are normalized by the expansion then $a_{\text{up}} = -\pi i \gamma$. The normalized horn map is defined on the unit disk and its singular values are real. For the parabolic renormalization, we need first to semi-conjugate by E , which gives a map fixing 0 with multiplier $e^{2\pi^2 \gamma}$. Then we pre and post compose by two linear maps so that the derivative at the origin is 1. The claim follows.

We mentioned earlier that $\gamma[B_d] = \frac{3}{20} \cdot \frac{d^2+1}{d^2-1}$. The ratio $|\nu|/r$ thus ranges between $\approx 1/140$ (for $d = 2$) and $\approx 1/20$ (for $d = \infty$). In the case $d < \infty$, notice that there is a tiny loop bounding a small yellow box containing the origin and that looks like a droplet. When d increases the angle at the tip of the loop decreases and the tip gets closer to the boundary of the domain of definition of the map. In the case $d = \infty$, the droplet touches the boundary. Now every blue box is mapped to the set $|z| > |\nu|$ as a universal cover, the yellow box containing the origin is mapped 1 : 1 to $|z| < |\nu|$ and every other yellow box is mapped as a universal cover to $|z| < |\nu|$ minus the origin.

This partition of the domain of $\mathcal{R}[f]$ into two colors and the graph separating them is called the *structural chessboard* of $\mathcal{R}[f]$. It is different from the *dynamical chessboard* of $\mathcal{R}[f]$, which is defined only in the basin of its parabolic point $z = 0$. In particular the structural chessboard is *not* dynamically invariant. It is the image

Figure 14: Structure of $\mathcal{R}[B_3]$.Figure 15: Structure of $\mathcal{R}[B_\infty]$.

by E of the chessboard of h , subject to the same restriction, rescaling and possibly inversion, as were done to pass from h to $\mathcal{R}[f]$.

The next pictures illustrate Theorem 2. Figure 16 shows the famous case dubbed the Cauliflower: this is the Julia set of $z \mapsto z^2 + 1/4$. We removed the colors and drew the boundaries between boxes and the boundaries of the definition domains. The six images are ordered in a 2×3 rectangle whose first column figures the dynamical chessboard of f atop and of B_2 below. The next column represents views of their chessboards in repelling Fatou coordinates or, more precisely, two periods of their preimage by Ψ_{rep} . The last column is the projection to \mathbb{C}^* of the middle column by the map $E : z \mapsto \exp(2\pi iz)$. The vertical arrows are isomorphisms between the three pairs of domains, mapping graph to graph, respecting box colors (not figured here) and even better: they are structure isomorphisms for the following respective maps (properly normalized): the attracting Fatou coordinate for the first column, the horn map for the second column and the parabolic renormalization for the last one. This diagram is also commutative if one adds the following self maps of the six sets: column 1: f, B_2 , column 2: T_1, T_1 , column 3: Id, Id . From all this we can build a big commutative diagram, but we do not think that it would not be much readable. Note that the tiny loops in the last column are the images of the big unbounded square that lie above in both middle images. The image of this square by Ψ_{rep} is one of the two f -invariant (resp. B_2 -invariant) squares (they touch the fixed point), but the latter has many other preimages by Ψ_{rep} .

Figure 17 shows the analog, but for the exponential map $z \mapsto e^z - 1$. The caption of Figure 7 gives more explanation about the color scheme of the top row. One thing worth noticing in the middle top image: *all* the yellow and blue components, not only the topmost, are unbounded (each has countably many arms that extend to the right, in some of the channels between hairs of the black set.) The image in the upper right corner, looking like a yin yang symbol, is very interesting, but we need to zoom near the center to see the details: this is done on Figure 18.

2.4. Inou and Shishikura's sub-structure. To finish this visualization chapter, we present here the sub-structure and how it fits within the picture for B_2 .

The first set of drawings shows one of the ways Inou and Shishikura used to present it. They defined a Riemann surface with a natural projection over \mathbb{C}/\mathbb{Z} as follows: cut the cylinder \mathbb{C}/\mathbb{Z} so as to retain only the part where $\text{Im}(z) > -\eta$ with $\eta = 2$.⁽¹³⁾ Slit this cylinder along the vertical segment from 0 to $-\eta i$. To this, glue the rectangle $\text{Re}(z) \in]-1, 1[$ and $\text{Im}(z) \in]-\eta, \eta[$, cut along the same segment. As usual with Riemann surfaces, we glue each side of the segment in one piece to the opposite side on the other piece. This is represented on the upper left part of Figure 19. This method is reminiscent of the way Perez-Marco uses to build structures in his work. Below it in the same figure, is a tentative to picture the way it projects to the cylinder \mathbb{C}/\mathbb{Z} , while on its right there is a planar open set isomorphic to it (conformal moduli are not respected in the figure). In the lower right corner, there is the image of the lower left by $z \mapsto \exp(2\pi iz)$ (rotated by 90 degrees). The right column is a map f with structure \mathcal{B} . The left part of Figure 20 accurately shows how \mathcal{B} sits as a substructure of the structure of $\mathcal{R}[B_2]$. The right part identifies the pieces.

¹³There is some flexibility in the value of this lower bound, in [IS04], they proved that their theorem holds for any real η between 13 and 2 included. Here, we drew the domain only for their original value $\eta = 2$.

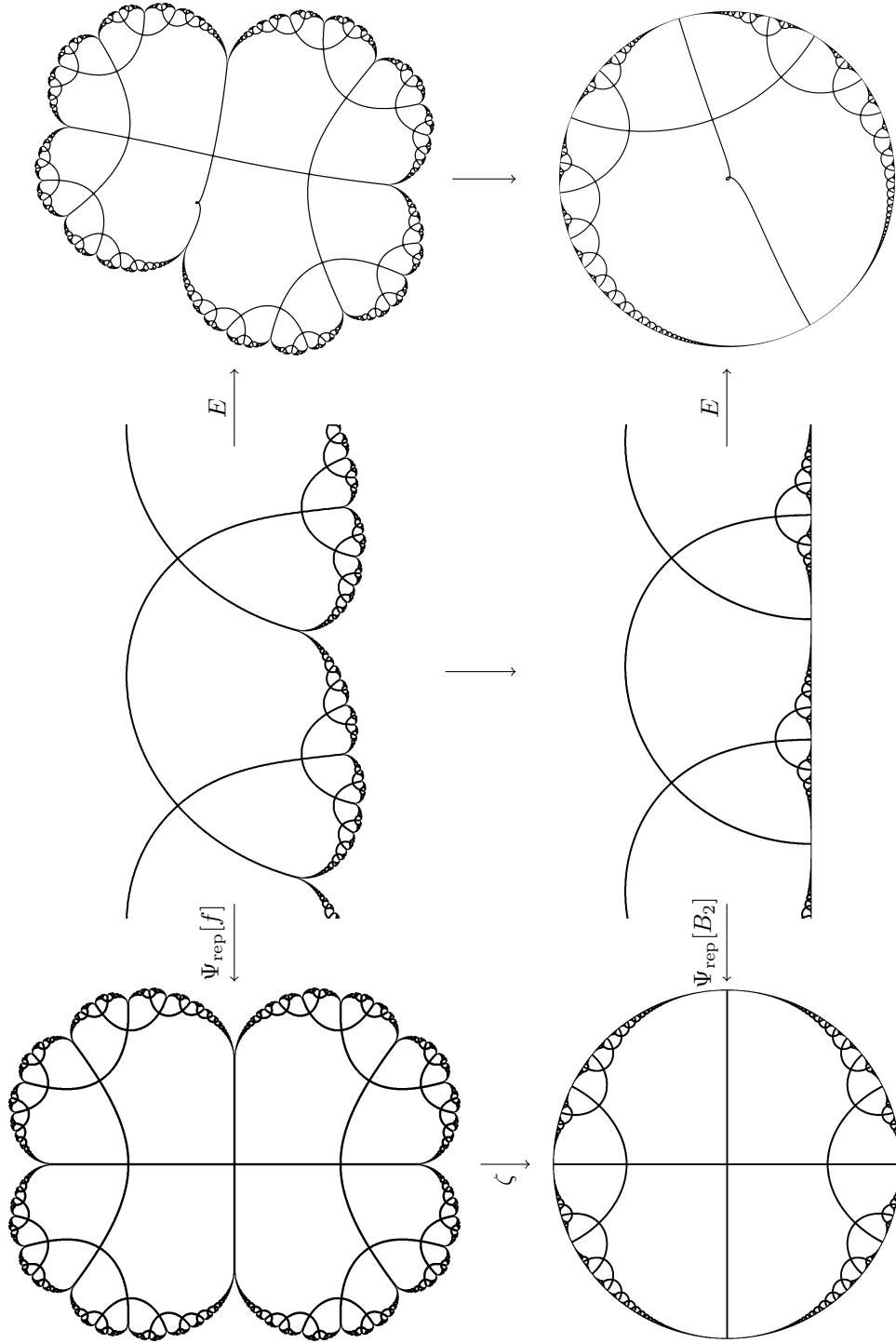


Figure 16: (rotated 90°) Illustration of Theorem 2 for $f(z) = z^2 + 1/4$. See the text page 27 for a description.

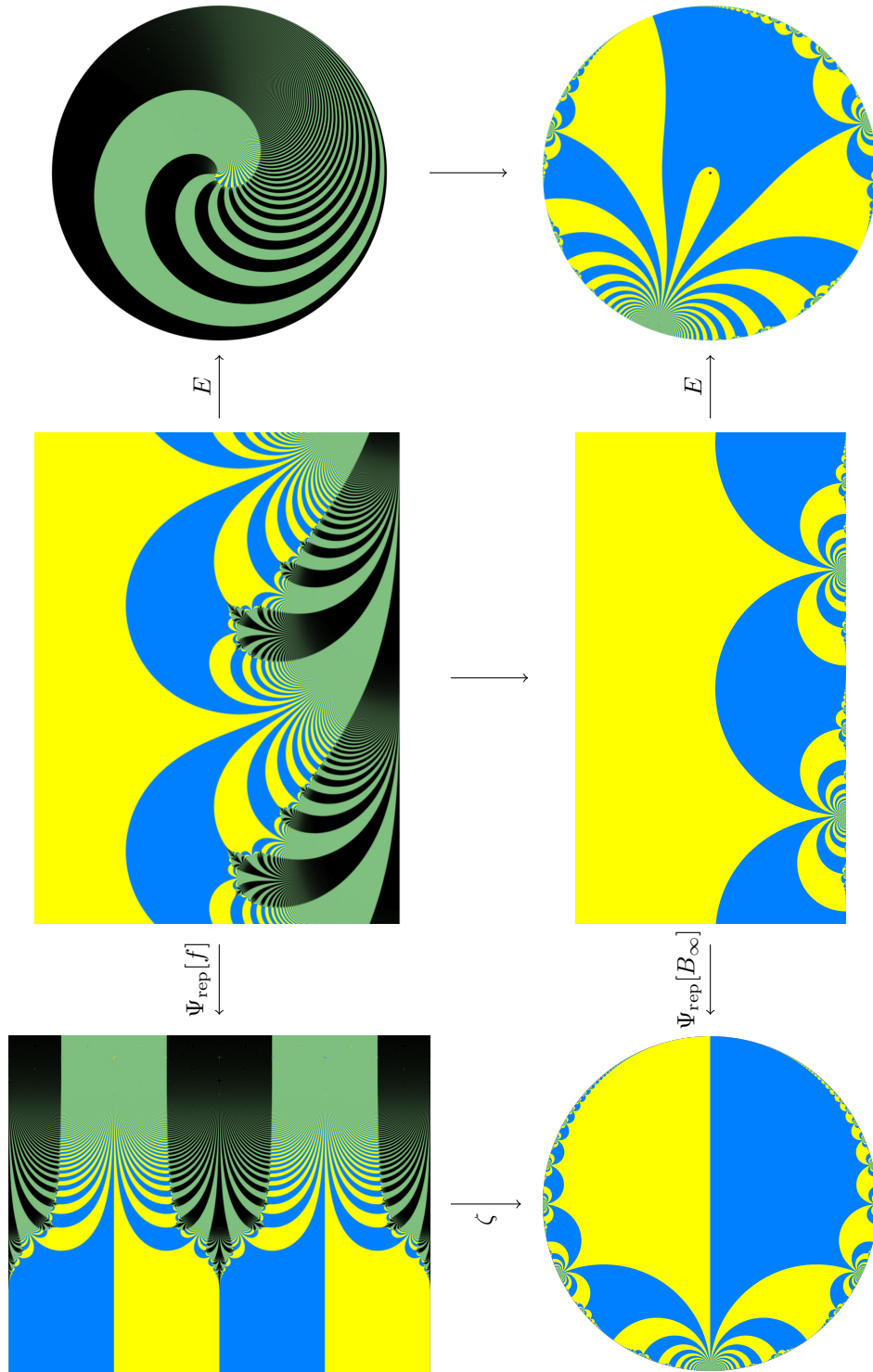


Figure 17: (rotated 90°) Analog of Figure 16 for $f(z) = e^z - 1$ and B_∞ . See the text for a description. There are enlargements of the top right image on Figure 18.

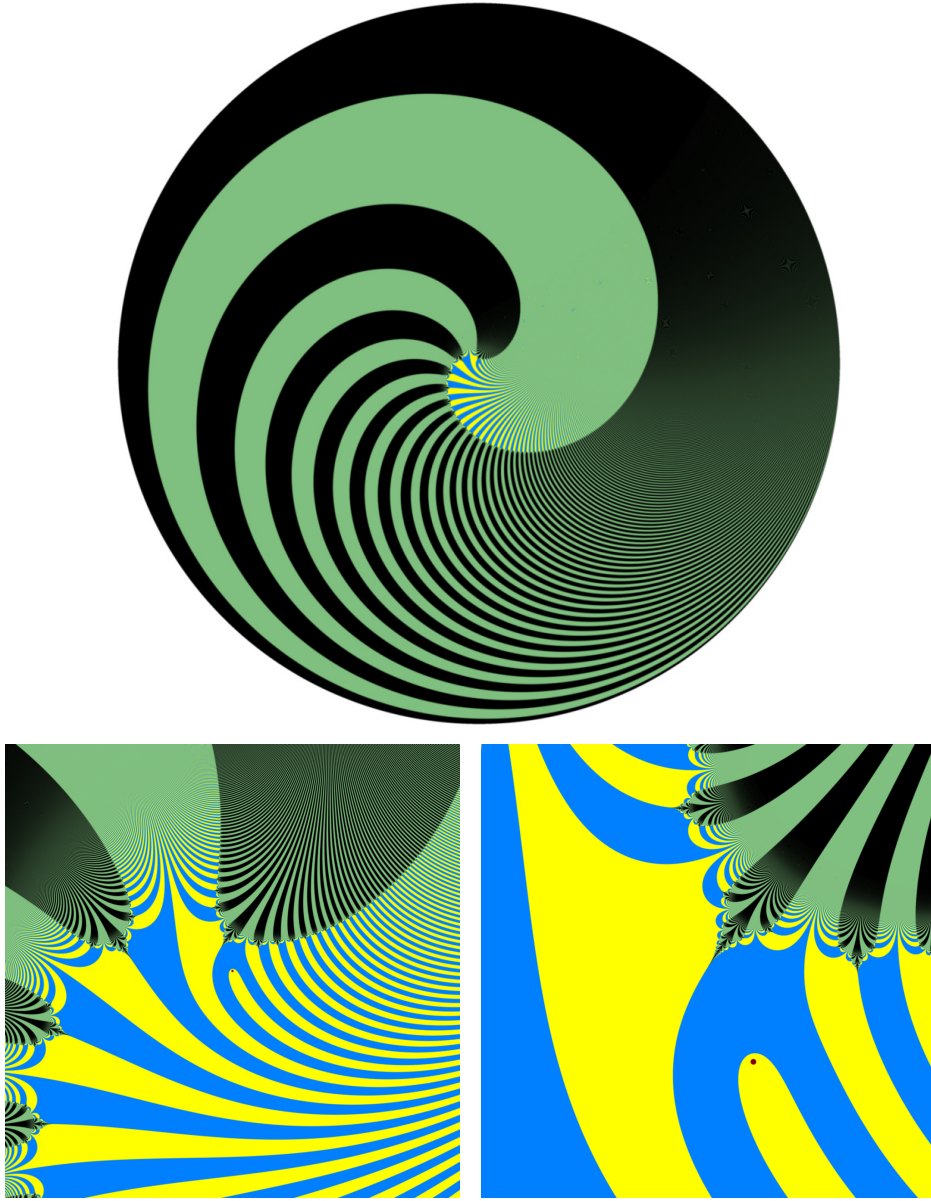


Figure 18: Zoom on the origin, for the picture of the structure of the upper parabolic renormalization of the parabolic exponential map $f(z) = e^z - 1$. The domain of definition of $\mathcal{R}[f]$ is the complement of the black set. The closure of this domain turns out to be a euclidean disk. The complement of this domain is sort of a cantor bouquet that winds infinitely many times as one approaches the boundary of the disk. On the top picture, the structure is too fine to be properly seen. Below, we added two closer looks near the origin, pinpointed by a purple dot.

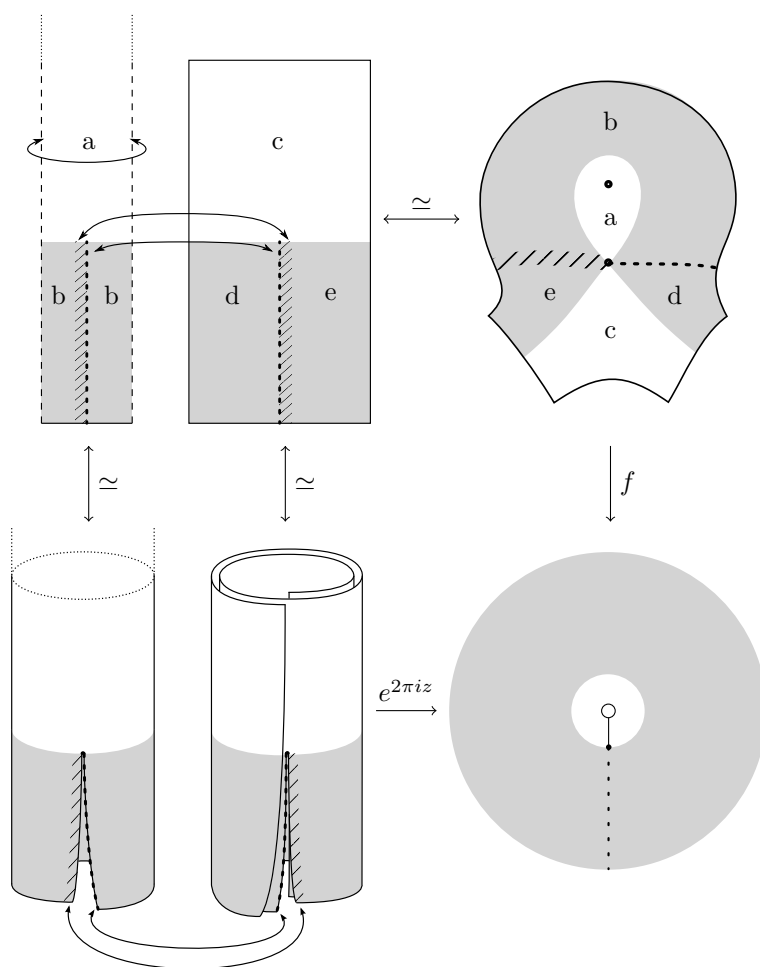


Figure 19: Only the upper left section of this figure is conformally correct. Explanations in the text on page 27.

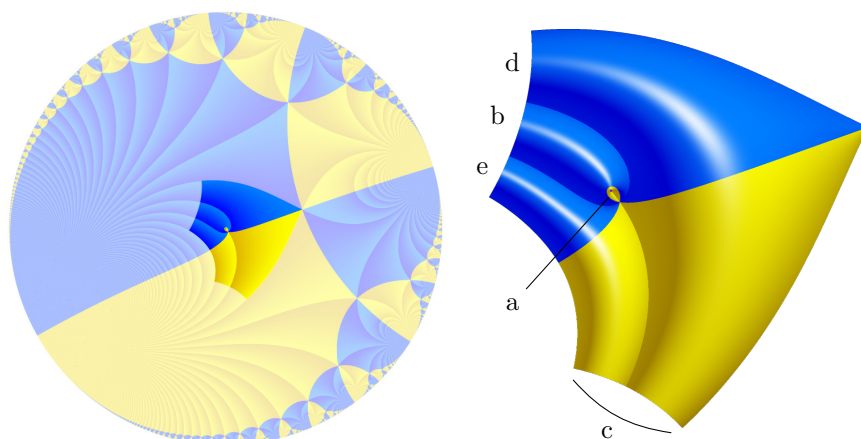


Figure 20: Caption in the text. Note that compared to the upper right part of Figure 19, there is a supplementary corner. The picture is accurate.

Structure \mathcal{A} is a substructure of \mathcal{B} , obtained by mapping conformally the domain of f minus the origin to the complement of the closed unit disk and removing the interior of some specific and explicitly defined ellipse (see [IS04]). The result, mapped to the set of Figure 20, is shown on Figure 21.

Let us call D the domain of $\mathcal{R}[B_d]$: this is a disk centered on the origin; Let $U \Subset V \Subset D$ be the sub-domains corresponding to respectively \mathcal{A} and \mathcal{B} . Inou and Shishikura worked with the particular sets we just described. It is more natural, though not easy, to take for U and V a pair of disks centered on the origin. The objective of the present article is to prove that this works. The downside is that we lose unicriticality of maps in the class we construct. Yet, it still applies to unicritical polynomials, after taking one renormalization (they become multicritical, with only one critical value); recall that even Inou and Shishikura need to take first one iteration of renormalization of to get a map in their class from a quadratic polynomial anyway. The upside is that our approach will work for critical points of any degree.

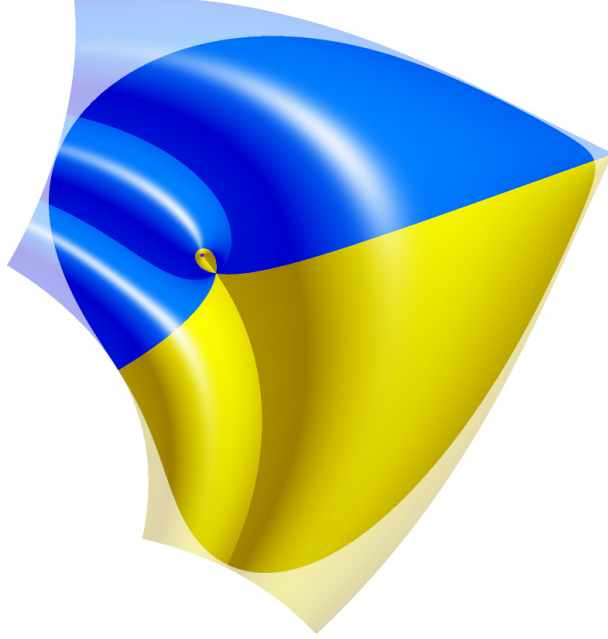


Figure 21: Comparison of \mathcal{A} and \mathcal{B} . The picture is accurate. Though it is hard to see, the boundary of the light-toned domain and the boundary of the color-saturated domain are disjoint.

3. PROOF

The element of hyperbolic metric of a connected open subset U of \mathbb{C} will be denoted by $\rho_U(z)|dz|$ and the corresponding hyperbolic distance by $d_U(z, z')$.

3.1. A convenient notation. Given $r \in]0, 1[$ and a subset U of \mathbb{C} conformally equivalent to \mathbb{D} and containing 0, we will denote

$$U \odot r = \{z \in U \mid d_U(0, z) < d_{\mathbb{D}}(0, r)\}.$$

Note that $U \odot r = \phi(B(0, r))$ where $\phi : \mathbb{D} \rightarrow U$ is a conformal isomorphism mapping 0 to 0.

Recall that we denoted $E(z) = e^{2\pi iz}$, which is a universal cover from \mathbb{C} to \mathbb{C}^* . Given a set of the form $V = E^{-1}(U)$ where U is as above, we will denote

$$V \Vdash r = E^{-1}(U \odot r).$$

3.2. Outline. Our main theorem will be proved in two steps. Let us fix in this section some $d \in \mathbb{N}$ with $d \geq 2$. From now on, parabolic renormalization refers to upper parabolic renormalization. We want to denote $\mathcal{R}[B_d]$ the parabolic renormalization of the Blaschke map B_d , whose importance was explained in the introduction (see Section 1.5). This is a slight abuse of notation since we defined \mathcal{R} for maps with only one attracting petal whereas B_d has two. In Section 1.5 we defined the objects $\Phi_{\text{attr}}[B_d]$, $\Psi_{\text{rep}}[B_d]$ and $h[B_d]$. We define $\mathcal{R}[B_d]$ as the semi-conjugate of $h[B_d]$ by E . In Section 2.2 we introduced a semi-conjugate C_d of B_d by a 2:1 rational map, such that C_d has only one attracting petal, and we gave relations between the objects for B_d and the objects for C_d . Note that $\mathcal{R}[B_d]$ coincides with $\mathcal{R}[C_d]$. The domain of $\mathcal{R}[B_d]$ is disk of center 0 and radius depending on normalizations. We will work with a normalization such that $\mathcal{R}[B_d]$ is defined on the unit disk, see the forthcoming Section 3.3.

Let

$$\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : \mathbb{D} \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2)\}$$

and

$$\mathcal{F}_\varepsilon = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : B(0, 1 - \varepsilon) \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2)\}.$$

In other words, \mathcal{F} is Shishikura's invariant class consisting of maps structurally equivalent to the renormalization of the Blaschke product B_d , and \mathcal{F}_ε is a class of maps having only a subset of this structure. The smaller ε , the more structure. Note that $\mathcal{F} = \mathcal{F}_0$. To be more precise and to stick to the language introduced in Section 1.1, let I be a singleton. If we mark the origin by the unique map $I \rightarrow \{0\}$, maps in \mathcal{F}_ε are all $(I, \widehat{\mathbb{C}})$ -structurally equivalent. We will prove the following more precise version of the main theorem (page 12):

Theorem. *The main theorem holds with \mathcal{B} = the structure of maps $f \in \mathcal{F}_{\varepsilon_1}$ with marked point 0 and \mathcal{A} = the substructure $\mathcal{F}_{\varepsilon_0}$, for some pair $\varepsilon_0 > \varepsilon_1$.*

The class of Schlicht maps is denoted \mathcal{S} , thus $\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi \in \mathcal{S}\}$. The two steps are the following:

- (1) Contraction: for $f \in \mathcal{F}$ denote $f = \mathcal{R}[B_d] \circ \phi_1^{-1}$, $\phi_1 \in \mathcal{S}$. Then by Theorem 2, with an appropriate normalization, $\mathcal{R}[f]$ is of the form $\mathcal{R}[B_d] \circ \phi_2^{-1}$, $\phi_2 \in \mathcal{S}$. We will prove that “the definition of $\mathcal{R}[f]$ on $\text{Dom}(\mathcal{R}[f]) \odot (1 - \varepsilon)$ uses only iteration of f on $\text{Dom}(f) \odot (1 - \varepsilon')$ where $\varepsilon' \gg \varepsilon$ ”.
- (2) Perturbation: for a map $f \in \mathcal{F}$, we will define a continuous deformation $f_t \in \mathcal{F}_t$. Every map in \mathcal{F}_t will be a deformation of a map in \mathcal{F} . We will prove that $\mathcal{R}[f_t]$ has structure at least \mathcal{F}_ε , provided $t \leq \varepsilon'/K$ for some $K > 1$, where ε' is given by the first step.

Let us give a slightly more detailed formulation of these two steps; we leave here some imprecisions; they will be fully stated and proven in details in Sections 3.6 to 3.9.

Step 1: Let $E(z) = e^{2\pi iz}$, Φ_{attr} the extended attracting Fatou coordinate of f , Ψ_{rep} the extended repelling inverse Fatou coordinate of f , both normalized according to Theorem 2, and recall that $\mathcal{R}[f](z)$ can be defined (up to pre and post composition by two linear maps) as

$$E(\Phi_{\text{attr}}(f^m(\Psi_{\text{rep}}(u)))),$$

where $u \in E^{-1}(z)$ is chosen so that it belongs to the image of the repelling petal by the repelling Fatou coordinates and $m \in \mathbb{N}$ is chosen so that $f^m(\Psi_{\text{rep}}(u))$ belongs to the attracting petal. So we are following the orbit of $w = \Psi_{\text{rep}}(u)$ under iteration of f from the repelling petal to the attracting petal. The claim is that this orbit stays in $\text{Dom}(f) \odot 1 - \varepsilon'$. Now recall that by the properties of the extended repelling Fatou coordinates, we have $f^k(w) = \Psi_{\text{rep}}(u+k)$ and that the domain of definition of Ψ_{rep} is invariant by the translation T_1 . Therefore, using that $E^{-1}(\text{Dom } \mathcal{R}[f] \odot 1 - \varepsilon)$ is equal to the translate by an appropriate complex constant of the domain of the horn map h , point (1) above can be stated as follows:

$$\Psi_{\text{rep}}(\text{Dom}(h) \vdash 1 - \varepsilon) \subset \text{Dom}(f) \odot 1 - \varepsilon'.$$

The relation $\varepsilon' \gg \varepsilon$ will take the form:

$$\log \frac{1}{\varepsilon'} \leq c' + c \log \left(1 + \log \frac{1}{\varepsilon} \right)$$

for some positive constants c, c' (Proposition 22).

Step 2: In the perturbation part, given $r = 1 - t_0$ and $f \in \mathcal{F}_r$, we define an element $f_0 \in \mathcal{F}$ together with a smooth interpolation f_t , $t \in [0, t_0]$, between f_0 and $f = f_{t_0}$. It has the following form:

$$f_t(z) = \mathcal{R}[B_d] \circ \phi_t^{-1}.$$

The map ϕ_t is a univalent map, defined on $B(0, 1 - t)$ with $\phi_t(0) = 0$ and $\phi_t'(0) = 1$ and is defined as follows: let $r_t = 1 - t$, decompose $f(z) = \mathcal{R}[B_d] \circ \phi^{-1}$, let $\phi(z) = r_{t_0}^{-1} \tilde{\phi}(r_{t_0} z)$, whence $\phi \in \mathcal{S}$, and define

$$\phi_t(z) = r_t \phi(r_t^{-1} z).$$

Note that ϕ_t is *not* the restriction of ϕ to $B(0, 1 - t)$; in fact there are plenty of univalent maps $\tilde{\phi}$ on $B(0, r)$ that are not the restriction of a univalent map defined on $B(0, 1)$.

Now since $f_0 = \mathcal{R}[B_d] \circ \phi^{-1}$ belongs to \mathcal{F} , its renormalization $\mathcal{R}[f_0]$ decomposes as $\mathcal{R}[B_d] \circ \phi_2^{-1}$ for some Schlicht map ϕ_2 . By the first step, given $\varepsilon > 0$ and a point $z \in \text{Dom } \mathcal{R}[f_0] \odot (1 - \varepsilon) = \phi_2(B(0, 1 - \varepsilon))$, we know that the value of $\mathcal{R}[f_0]$ is obtained through iteration under f_0 of a point w in the repelling petal of f_0 , point whose orbit remains in $\text{Dom } f_0 \odot 1 - \varepsilon' = \phi(B(0, 1 - \varepsilon'))$ with $\varepsilon' \gg \varepsilon$. We will then vary t from 0 to t_0 and follow by continuity the points in the orbit of w , not by fixing the initial value, but instead by imposing that their attracting Fatou coordinate stays the same, where we normalize the attracting Fatou coordinates (it varies with t since f_t does) by putting its critical values at the nonnegative integers. In particular, w moves with t . A local study shows that the tail of the orbit will not move much. The motion of the other points will be bounded from above inductively by iterating backwards along the orbit, until we reach w . We will measure the motion in terms of the hyperbolic metric on the complement in \mathbb{C} of the post-critical orbit of f_0 . The study will show (Proposition 43) that there is some $K > 0$ independent of f (necessarily $K > 1$) such that, provided ε' is small enough, an orbit that is initially completely contained in $\text{Dom}(f_0) \odot 1 - \varepsilon'$ survives

all the way as t varies from 0 to ε'/K . Thus $\mathcal{R}[f_t]$ has at least structure $\overline{\mathcal{F}_\varepsilon}$ provided $t \leq \varepsilon'/K$. The main theorem thus holds for $\mathcal{A} = \overline{(0, f \in \mathcal{F}_{\varepsilon_0})}$ and $\mathcal{B} = \overline{(0, f \in \mathcal{F}_{\varepsilon_1})}$ with $\varepsilon_0 = \varepsilon'/K$ and $\varepsilon_1 = \varepsilon$ with ε small enough, as $\varepsilon' \gg \varepsilon$ will imply $\varepsilon_0 > \varepsilon_1$.

3.3. Normalizations. In the rest of Section 3, i.e. in the proof of the main theorem,

- normalized Fatou coordinates refer to the normalization by the asymptotic expansion at infinity, convention numbered 2 on page 7,
- Φ_{attr} will refer to extended attracting Fatou coordinates, normalized according to the same convention,
- Ψ_{rep} will refer to extended inverse of the repelling Fatou coordinates that are normalized according to the same convention,
- $h_{\text{nor}} = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$,
- $\mathcal{R}[f]$ is the parabolic renormalization, normalized by the critical value (convention numbered 3 on page 8); see details below
- in the second part, Ψ_t and Φ_t will denote the extended repelling/attracting inverse/direct Fatou coordinates of f_t , normalized according to a convention analog to number 3.

Let f satisfy the hypotheses of Theorem 2, so that $\mathcal{R}[f]$ is, up to normalization, equal to $\mathcal{R}[B_d] \circ \phi^{-1}$ for some $\phi \in \mathcal{S}$. Let us call (only in this paragraph) U the connected component of $\text{Dom}(h_{\text{nor}})$ that contains an upper half plane and Ξ the map such that

$$E \circ h_{\text{nor}}|_U = \Xi \circ E.$$

Then $\mathcal{R}[f] = M_a \circ \Xi \circ M_b^{-1}$ for a pair of linear maps $M_a : z \mapsto az$ and $M_b : z \mapsto bz$ that depend on f , hence

$$(1) \quad M_a \circ E \circ h_{\text{nor}}|_U = \mathcal{R}[f] \circ M_b \circ E.$$

Let us first apply this to $f = C_d$ (recall that we defined $\mathcal{R}[B_d] = \mathcal{R}[C_d]$). The map C_d commutes with $z \mapsto \bar{z}$, and its Julia set is the positive real axis. As a consequence, with the normalization above, $\Psi_{\text{rep}}[B_d]$ commutes with $z \mapsto \bar{z}$, and $h_{\text{nor}}[B_d]$ is defined on the complement of the real line and commutes too with $z \mapsto \bar{z}$. We now choose to normalize $\mathcal{R}[B_d]$ so that $b = 1$, ⁽¹⁴⁾ whence

$$M_a \circ E \circ h_{\text{nor}}[B_d]|_{\mathbb{H}} = \mathcal{R}[B_d] \circ E.$$

In particular, $\mathcal{R}[B_d]$ is defined on the unit disk:

$$\text{Dom } \mathcal{R}[B_d] = \mathbb{D}.$$

Theorem 2 states that for a map f satisfying the hypotheses in there exists a choice of a and b in (1), such that

$$\mathcal{R}[f] = \mathcal{R}[B_d] \circ \phi^{-1}$$

i.e. such that $\mathcal{R}[f] \in \mathcal{F}$. In particular $\mathcal{R}[f]$ and $\mathcal{R}[B_d]$ have the same (unique) critical value. Hence this choice of a and b is a normalization by the critical value, convention numbered 3 on page 8. The class \mathcal{F} is stable by renormalization with this convention:

$$\mathcal{R} : \mathcal{F} \rightarrow \mathcal{F}.$$

For reference, let us state here the following version of universality

¹⁴This implies $a = e^{-2\pi^2\gamma[B_d]}$ because the derivative of $\mathcal{R}[B_d]$ is 1 at the origin, but we will not use that fact.

Lemma 6. *For $f \in \mathcal{F}$, let v_f denote the critical value of f and $v'_f = \Phi_{\text{attr}}(v_f)$. There is a conformal map ϕ from the upper component $U[B_d]$ of $\text{Dom}(h_{\text{nor}}[B_d])$ to the upper component $U[f]$ of $\text{Dom}(h_{\text{nor}}[f])$ that commutes with T_1 and such that*

$$T_\tau \circ h_{\text{nor}}[f]|_{U[f]} = h_{\text{nor}}[B_d] \circ \phi^{-1}$$

with $\tau = v'_{B_d} - v'_f$.

Proof. By Corollary 3, $\Phi_{\text{attr}}[B_d] \circ \zeta = \tau + \Phi_{\text{attr}}[f]|_A$ for some $\tau \in \mathbb{C}$ and $\zeta : A \rightarrow \mathbb{D}$ the conjugacy from f on its immediate parabolic basin to B_d . By applying to the unique critical value of $f|_A$ we get $\tau = v'_{B_d} - v'_f$. By the complement after Theorem 2, $\Psi_{\text{rep}}[B_d] \circ \phi^{-1} = \zeta \circ \Psi_{\text{rep}}[f]$ for some conformal isomorphism $\phi : U[B_d] \rightarrow U[f]$ that commutes with T_1 . We conclude using $h = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$. \square

3.4. Chessboards. Just before we begin the proofs, let us recall that maps $f \in \mathcal{F}$ have a *structural chessboard* and a *dynamical chessboard*. The first is a partition of $\text{Dom } f$ that is a pre-image of the partition of \mathbb{C}^* cut by the circle of center 0 and passing through the critical value of f . The second is a partition of the basin (or of the immediate basin) of the parabolic point $z = 0$ of f , and is f -invariant. The second is also a structural object w.r.t. $\Phi_{\text{attr}}[f]$. See Section 2.3 for more details.

We defined a chessboard for the horn maps h associated to parabolic points of maps $f \in \mathcal{F}$ (more generally to maps f satisfying the hypotheses of Theorem 2). It is the preimage in repelling Fatou coordinates of the dynamical chessboard of f and it is also the preimage by h of the partition of its range cut by a horizontal line. There is a box that contains an upper half plane, we call it the *main upper box* of h . Similarly the box that contains a lower half plane is called the *main lower box* of h .

3.5. Toolkit. In this section we redo classical computations on Fatou coordinates and first terms of their expansion. We add dependence on a map staying in a compact class and put the emphasis on uniformity of the bounds obtained. The section mainly serves as a reference for the rest of the text. The trusting reader may skip it.

Proposition 7. *Assume \mathcal{G} is a set of holomorphic maps $g : \mathbb{D} \rightarrow \mathbb{C}$ with $g(z) = z + c_g z^2 + \dots$, that \mathcal{G} is compact for the topology of local uniform convergence and that c_g is never 0, i.e. that g has one attracting petal. Denote γ_g the iterative residue of g . Let \log_p be the principal branch of the complex logarithm. Then there exists r_0 such that $\forall g \in \mathcal{G}$*

- the disk D_{attr} of diameter $[0, r_0 e^{i\alpha}]$ where α is the direction of the attracting axis of g , is contained in the parabolic basin of g ; $g(D_{\text{attr}}) \subset D_{\text{attr}}$ and every orbit in the parabolic basin eventually enters D_{attr} ;
- the extended attracting Fatou coordinate ϕ_g of g is injective on D_{attr} and $\phi_g(D_{\text{attr}})$ is of the form $\{z \in \mathbb{C} \mid \text{Re}(z) > \zeta(\text{Im}(z))\}$ with $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ an analytic function satisfying $\zeta(x)/x \rightarrow 0$ when $x \rightarrow \pm\infty$;
- on D_{attr} , the normalized attracting Fatou coordinates Φ of g and the map $\tilde{\Phi} : z \mapsto \frac{-1}{c_g z} - \gamma_g \log_p \frac{-1}{c_g z}$ have a difference uniformly bounded by a quantity that is independent of g .

The above points also hold if r_0 is replaced by any smaller positive real.

Proof. The techniques in this proof are standard (see [Lav89], [DH85], [Shi00], [Ché08]). We will insist here on providing uniformity of the bounds as g varies in \mathcal{G} .

By compactness, uniformly on \mathcal{G} :

- c_g is bounded away from 0: $\exists \varepsilon > 0$ such that $\forall g \in \mathcal{G}$, $|c_g| \geq \varepsilon$;

- g is bounded on $B(0, 1/2)$: $\exists K > 0$ such that $\forall g \in \mathcal{G}$, $|g| \leq K$ on $B(0, 1/2)$.

Also, by Cauchy's inequality,

$$|c_g| \leq 4K.$$

Since $|g(z) - z| \leq K + 1/2$ on $B(0, 1/2)$, we get $|g(z) - z| \leq K'|z|^2$ with $K' = 4K + 2$, and in particular g does not vanish on $B(0, 1/K')$ except at the origin.

We will make a series of change of variables $z \mapsto u \mapsto w \mapsto \xi$ with

$$u = \frac{-1}{c_g z}, \quad w = u - \gamma_g \log_p u, \quad \xi = \Phi(z)$$

Where \log_p denotes the principal branch of the logarithm. We will denote $z' = f(z)$ and use the notation $u \mapsto u', \dots, \xi \mapsto \xi'$ for the dynamical systems $z \mapsto z'$ will be conjugated to.

The first change of variable is injective on \mathbb{C}^* . It maps D_{attr} to the half plane $H_{\text{attr}} : \text{Re}(u) > U_0(g) = 1/r_0|c_g|$. An asymptotic expansion gives

$$u' = u + 1 + \frac{\gamma_g}{u} + \mathcal{O}(u^{-2})$$

The condition $z \in B(0, 1/K')$ becomes $|u| > K'/|c_g|$. Under this condition the map $u \mapsto u'$ is holomorphic, and depends continuously on g . From compactness of \mathcal{G} , it follows that these restrictions form a compact family too. In particular, if we further restrict to $|u| > 1 + K'/|c_g|$, we get by a simple application of the maximum principle that

$$\begin{aligned} |u' - (u + 1)| &\leq M_1/u \\ |u' - (u + 1 + \frac{\gamma_g}{u})| &\leq M_2/u^2 \end{aligned}$$

for some constants M_1, M_2 independent of $g \in \mathcal{G}$. Thus for $r_0 \leq 1/(|c_g| \max(1 + K'/|c_g|, 4/M_1))$, we have

$$\frac{M_1}{|u|} \leq \frac{1}{4}$$

thus

$$|u' - (u + 1)| \leq 1/4$$

thus the set H_{attr} is invariant under the dynamics of $u \mapsto u'$, so D_{attr} is invariant under $z \mapsto z'$. It is also easy to see that in the u -coordinate, an orbit tending to ∞ must eventually get into H_{attr} . The right hand side of the condition on r_0 depends continuously on g and reaches thus a positive minimum: it is satisfied as soon as $r_0 \leq r_1$ where r_1 is independent of g .

The constant γ_g is finite and depends continuously¹⁵ on g . Thus it is bounded over \mathcal{G} , say by Γ :

$$|\gamma_g| \leq \Gamma.$$

The change of variable $w = u - \gamma_g \log_p u$ has derivative $1 - \gamma_g/u$. It is thus injective on the convex set $\text{Re}(u) > 2|\gamma_g|$. Thus when $r_0 \leq r_2$ where $r_2 = \min_{g \in \mathcal{G}} (1/2|\gamma_g c_g|) > 0$, then $\forall g \in \mathcal{G}$, the map $u \mapsto w$ is injective on H_{attr} . We will require in fact a bit more: $r_0 \leq r'_0 = r_2/2$, so that $|\frac{\partial w}{\partial u} - 1| \leq \frac{1}{4}$. This implies that the image of H_{attr} is a set that is of the form $\text{Re}(w) > \zeta(\text{Im}(w))$ for some analytic function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ that depends on g and satisfies $|\zeta'(y)| < 1/\sqrt{15}$. Moreover, $\zeta(y)/y \rightarrow 0$ when $y \rightarrow \pm\infty$ because $w \sim u$ when $|u| \rightarrow \infty$. In this new coordinates, we get

$$w' - w = \int_{[u, u']} \left(1 - \frac{\gamma_g}{a}\right) da$$

¹⁵because it is equal to $1 - a_3/c_g^2$ if we denote $g(z) = z + c_g z^2 + a_3 z^3 + \dots$

whence

$$w' - w = 1 + \frac{\gamma_g}{u} + \frac{\leq M_2}{u^2} - \gamma_g \log \left(1 + \frac{1}{u} + \frac{\leq M_1}{u^2} \right)$$

where $\leq M_2$ means a complex number that depends on u but whose module is at most M_2 ; we require $r_0 \leq r_3$ where r_3 is chosen independent of g and so that the quantity $\frac{1}{u} + \frac{\leq M_1}{u^2}$ has necessarily modulus $< 1/2$: recall that $1/u = -c_g z$ and that $|c_g| \leq 4K$. We can then apply the following estimate: $|a| < 1/2 \implies |\log(1+a) - a| \leq L_0 |a|^2$ for some $L_0 > 0$. Hence (thanks to a cancellation of the term γ_g/u)

$$w' - w = 1 + \frac{\leq M_2}{u^2} + \gamma_g \frac{\leq M_1}{u^2} + \gamma_g \frac{\leq (1+1/4)^2 L_0}{u^2}$$

(recall that $M_1/|u| < 1/4$). Thus for some constant M_3 independent of g :

$$|w' - (w+1)| \leq \frac{M_3}{u^2}.$$

The Fatou coordinates can be defined by

$$\Phi(z) = \mu + \lim(w_n - n)$$

where μ is a constant (that depends on the normalization) and w_n is the n -th iterate of w under the dynamics. Since $\operatorname{Re}(u_n) > \operatorname{Re}(u_0) + \frac{3}{4}n$ and $\operatorname{Re}(u_0) \geq \frac{1}{r_0|c_g|}$, using $r_0 \leq r_4 = \min(r_1, r'_2, r_3)$ we thus get

$$\lim |w_n - (w_0 + n)| \leq \sum \frac{M_3}{|u_n|^2} \leq \sum \frac{M_3}{\left(\frac{1}{4Kr_4} + \frac{3/4}{n}\right)^2} = M_4.$$

Thus $|\Phi(z) - (\mu + w)| \leq M_4$ holds on D_{attr} for all g . The normalizing constant μ is so that $\Phi(z) = w + o(1)$ as $z \rightarrow 0$ (iff. $w \rightarrow \infty$) and therefore $|\mu| \leq M_4$ whence: $\forall g \in \mathcal{G}, \forall z \in D_{\text{attr}},$

$$|\Phi(z) - w| \leq 2M_4.$$

Recall that H_{attr} is the image of D_{attr} in the u -coordinate and that it is equal to the half plane $\operatorname{Re}(u) > U_0(g) = 1/r_0|c_g|$. Let $U_4(g) = 1/r_4|c_g|$ and H_4 be defined by $\operatorname{Re}(u) > U_4(g)$. Let $\Theta : H_4 \rightarrow \mathbb{C}, u \mapsto \Phi(z)$. Then $|\Theta(u) - (u - \gamma_g \log_p(u))| \leq 2M_4$ and by Cauchy's inequality, $|\Theta'(u) - (1 - \gamma_g/u)| \leq 2M_4/(\operatorname{Re}(u) - u_4)$. In particular, the image of H_{attr} is of the form $\operatorname{Re}(z) > \zeta(\operatorname{Im}(z))$ for some function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ provided $r_0 \leq r_5 = r_4/(1 + 8M_4)$ so that $2M_4/(\operatorname{Re}(u) - u_4) \leq 1/4$ and provided $r_0 \leq r_6 = 1/16KT$ so that $|\gamma_g/u| \leq 1/4$. The fact that $\zeta(y)/y \rightarrow 0$ as $y \rightarrow \pm\infty$ follows again from $|\Theta(u) - (u - \gamma_g \log_p(u))| \leq 2M_4$.

We can now fix the value of r_0 to $\min(r_5, r_6)$ (or any smaller value) and this gives us a set D_{attr} that satisfies all points stated in the proposition. \square

Similar arguments provide:

Proposition 8. *Under the same assumptions as in the previous proposition, let*

$$D_{\text{rep}} = -D_{\text{attr}}.$$

Then for r_0 small enough the following holds: $\forall g \in \mathcal{G}$,

- *there is a branch ℓ of g^{-1} defined on a neighborhood of 0 containing D_{rep} such that $\ell(D_{\text{rep}}) \subset D_{\text{rep}}$, D_{rep} is contained in the parabolic basin of ℓ , every orbit in the parabolic basin of ℓ eventually enters D_{rep} ;*
- *a normalized repelling Fatou coordinate Φ_{rep} for g is defined on D_{rep} ; it is injective on this set and maps it to a domain of the form $\operatorname{Re}(z) < \zeta(\operatorname{Im}(z))$ for some analytic function ζ ;*

- $\Phi_{\text{rep}} - \tilde{\Phi}_{\text{rep}}$ is uniformly bounded on D_{rep} by a constant M_{rep} independent of g , where $\tilde{\Phi}_{\text{rep}} = \frac{-1}{c_g z} - \gamma_g \log_p \frac{1}{c_g z}$ (notice the change of sign inside the log compared to attracting Fatou coordinates).

We will also need a control on the inverse Fatou coordinates, that we easily deduce from the control on the Fatou coordinates:

Proposition 9. *Using the notations of Proposition 7, provided r_0 was chosen small enough, then for all $g \in \mathcal{G}$:*

- Let $\Psi = \Phi^{-1}$. Then the difference between $-1/c_g \Psi(Z)$ and $Z + \gamma_g \log Z$ is bounded by a quantity independent of g and of $Z \in \Phi(D_{\text{attr}})$.
- The domain of definition of Φ^{-1} , i.e. $\Phi(D_{\text{attr}})$, contains the set “ $\text{Re } Z > \xi(\text{Im } Z)$ ” where ξ is a function independent of g and satisfying $\xi(y) = \mathcal{O}(\log |y|)$ as $y \rightarrow \pm\infty$.

Proof. We will use the notations of the proof of Proposition 7. There was a change of variables $u = s(z) = -1/c_g z$ and a bound

$$|\Theta(u) - (u - \gamma_g \log_p u)| \leq M$$

for some constant M , where $\Theta(u) = \Phi(s^{-1}(u))$. Let us denote $p(Z) = \Theta^{-1}(Z) = s(\Psi(Z))$. Then

$$|Z - (p(Z) - \gamma_g \log_p p(Z))| \leq M.$$

There exists $C > 0$ such that for $|z| > C$ then $|\Gamma| \log_p |z| + M < |z|/4$ (recall $\Gamma = \sup_{g \in \mathcal{G}} |\gamma_g|$), whence if $r_0 < 1/C \sup |c_g|$ then H_{attr} is contained in $|u| > C$ and thus: $|\Theta(u) - u| < |u|/4$ i.e. $|\Theta(u)/u - 1| < 1/4$, i.e.

$$\forall Z \in \Phi(D_{\text{attr}}), |Z/p(Z) - 1| < 1/4.$$

Now

$$\begin{aligned} |p(Z) - (Z + \gamma_g \log_p Z)| &\leq |Z - (p(Z) - \gamma_g \log_p p(Z))| + |\gamma_g| |\log_p p(Z) - \log_p Z| \\ &\leq M + \sup |\gamma_g| \left| \log_p \frac{p(Z)}{Z} \right| \\ &\leq M + \sup |\gamma_g| \log \frac{4}{3}. \end{aligned}$$

The proof of the second point is similar. Recall that H_{attr} depends on g , and is defined by $\text{Re } z > a_g$ where $a_g = 1/r_0 |c_g|$. The image $\Phi(D_{\text{attr}}) = \Theta(H_{\text{attr}}) = \{z \in \mathbb{C} \mid \text{Re } z > \zeta(|\text{Im } z|)\}$ where $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ is an analytic function that depends on g . The map Θ extends to a neighborhood of the closure of H_{attr} and still satisfies $|\Theta(u) - (u - \gamma_g \log_p u)| \leq M$ on this closure. The curve $\zeta(\mathbb{R})$ is the image of ∂H_{attr} under this extension of Θ . Denote $x + iy = \Theta(a_g + ib)$ where $b \in \mathbb{R}$. Then $\log_p u = \log |u| + i \arg_p(u)$ and $\arg_p(u) < \pi/2$ thus the bound on $\Theta(u)$ yields for the real and imaginary parts:

$$\begin{aligned} |x - (a_g - \text{Re}(\gamma_g) \log |a_g + ib|)| &\leq M' := M + \Gamma \pi/2, \\ |y - (b - \text{Im}(\gamma_g) \log |a_g + ib|)| &\leq M'. \end{aligned}$$

There exists $C' > 0$, independent of g , such that for all $b \in \mathbb{R}$, $|\text{Im}(\gamma) \log_p |a_g + ib|| \leq |b|/2 + C'$. The second line thus yields $|b| \leq |b|/2 + C' + |y| + M'$ i.e. $|b| \leq 2|y| + M''$ for some M'' . Whence $x \leq \xi(y) := \sup(a_g) + M' + \Gamma \log |\sup(a_g) + i(2|y| + M'')|$, which is independent of g and has the right order of growth w.r.t y . \square

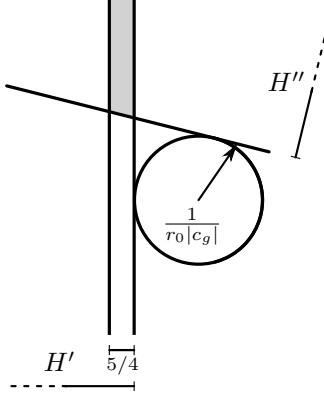
Proposition 10. *Under the same assumptions, there exists $h > 0$ such that for all $g \in \mathcal{G}$, the normalized extended repelling inverse Fatou coordinate Ψ_{rep} and the normalized extended horn map h_{nor} are defined on a set containing the half planes $\text{Im}(z) > h$ and $\text{Im}(z) < -h$, and injective on the union of those half planes.*

Moreover, for all $r > 0$, there exists $h > 0$ such that for all $g \in \mathcal{G}$, Ψ_{rep} maps these half planes inside the disk $B(0, r)$.

Proof. Let us continue with the notations of the proof of Proposition 7. Note that, provided r_0 has been chosen small enough, the maps $g \in \mathcal{G}$ are all injective on $B(0, r_0)$. Without loss of generality we assume $r < r_0$. Let us again work in the coordinates

$$u = s(z) = -1/c_g z.$$

Let $D'(r)$ be the disk of diameter $[0, re^{i\alpha}]$ where α is the repelling direction of f . (In particular $D_{\text{rep}} = D'(r_0)$.) The set $D'(r)$ is transformed into the half plane $H' : \text{Re}(z) < -1/r|c_g|$. To shorten formulas, we will work with $\Phi_{\text{rep}}^u(u) = \Phi_{\text{rep}} \circ s^{-1}(u)$, $\tilde{\Phi}_{\text{rep}}^u(u) = \tilde{\Phi}_{\text{rep}} \circ s^{-1}(u) = u - \gamma_g \log_p(-u)$ and $\Psi_{\text{rep}}^u(\xi) = s \circ \Psi_{\text{rep}}(\xi)$. Consider the line of slope $-1/\sqrt{15}$ that is tangent to the disk $B(0, 1/r|c_g|)$, outside which $|u' - (u + 1)| < 1/4$, and such that the open half plane H'' above this line does not contain the disk. Then H'' is stable: $u \in H'' \implies u' \in H''$. Consider now the vertical bi-infinite strip S of width $5/4$ whose rightmost bounding line is the boundary of H'_0 . Its image in repelling Fatou coordinates contains a fundamental domain for the translation $z \mapsto z + 1$. The intersection of S with H'' contains all points $u \in S$ with $\text{Im}(u) > h_1$ for some h_1 that depends on r and r_0 and the lower bound ε on $|c_g|$ mentioned at the beginning of the proof of Proposition 7. Using $|\Phi_{\text{rep}}^u - \tilde{\Phi}_{\text{rep}}^u| < M_{\text{rep}}$ and the upper bound $|\gamma_g| \leq \Gamma$, we deduce that $\Phi_{\text{rep}}^u(S \cap H'')$ contains every point of $\Phi_{\text{rep}}^u(S)$ with imaginary part $\geq h$, where h depends on the other constants but not on g .



Recall that Φ_{rep}^u maps the vertical line bounding H' to a y -graph, i.e. a curve which crosses each horizontal line exactly once. The translate by -1 of this curve is the image by Φ_{rep}^u of a curve C , preimage in H' of $\partial H'$ by $u \mapsto u'$. Because of the inequality $|u' - (u + 1)| < 1/4$, we get $C \subset S$. Thus $\Phi_{\text{rep}}^u(S)$ contains a domain bounded by a y -graph and its translate by -1 , i.e. a fundamental domain for the translation by -1 .

Let us prove that the domain of the extended normalized inverse repelling Fatou coordinate Ψ_{rep} contains all points at height $> h$. Recall Ψ_{rep} is defined by extending Φ_{rep}^{-1} , which is defined only on $\Phi_{\text{rep}}^u(H')$, by setting $\Psi_{\text{rep}}(\xi) = g^n(\Phi_{\text{rep}}^{-1}(\xi - n))$ for all $n \geq 0$ and all $\xi \in \mathbb{C}$ such that the right hand side is defined. Consider now $\xi \in \mathbb{C}$. By the fundamental domain property proved above, there exists $m \in \mathbb{Z}$ such that $\xi - m \in \Phi_{\text{rep}}^u(S)$. If $m \leq 0$ then $\xi \in \Phi_{\text{rep}}^u(H') = \text{Dom}(\Phi_{\text{rep}}^{-1})$ hence $\xi \in \text{Dom} \Psi_{\text{rep}}$. If $m \geq 0$ and $\text{Im}(\xi) > h$ then $\text{Im}(\xi - m) = \text{Im} \xi > h$ and thus we have seen that $u := (\Phi_{\text{rep}}^u)^{-1}(\xi - m)$ belongs to $H'' \cap S$. Since H'' is stable, the whole forward orbit of u belongs to H'' . In particular $g^m(\Phi_{\text{rep}}^{-1}(\xi - m))$ is defined, hence $\xi \in \text{Dom} \Psi_{\text{rep}}$. We have proven that the half plane “ $\text{Im}(\xi) > h$ ” is contained in $\text{Dom} \Psi_{\text{rep}}$.

Let now $\xi \in \mathbb{C}$ with $\text{Im}(z) > h$ and let us prove that $\Psi_{\text{rep}}(\xi) \in B(0, r)$ and to the parabolic basin. Again consider $m \in \mathbb{Z}$ such that $\xi - m \in \Phi_{\text{rep}}^u(S)$. Then in the case $m \geq 0$ we just saw that the whole orbit of u is in H'' , in particular the m -th iterate, which is equal to $\Psi_{\text{rep}}^u(\xi)$. Thus the point $\Psi_{\text{rep}}(\xi) = s^{-1}(\Psi_{\text{rep}}^u(\xi))$ belongs to $B(0, r)$. Also, the orbit of u tends to ∞ hence $\Psi_{\text{rep}}(\xi)$ belongs to the basin of the parabolic point of g . In the case $m \leq 0$, then $\xi \in \Phi_{\text{rep}}(D'(r))$ and thus $\Psi_{\text{rep}}(\xi) = \Phi_{\text{rep}}^{-1}(\xi) \in D'(r) \subset B(0, r)$. Since moreover $\xi - m$ satisfies the first case and thus the point $\Psi_{\text{rep}}(\xi - m)$ belongs to the parabolic basin, we get that $\Psi_{\text{rep}}(\xi)$

also belongs to the basin, as it is mapped to the former point by the $|m|$ -th iterate of g .

The proofs for the lower half plane “ $\text{Im } z < -h$ ” are similar. Let us prove injectivity of Ψ_{rep} on the union V of “ $\text{Im } z < -h$ ” and “ $\text{Im } z > h$ ”. First, it is injective on $U = \Phi_{\text{rep}}(D'(r))$, because it is equal to Φ_{rep}^{-1} there. The map g is injective on $\Psi_{\text{rep}}(V)$ because the latter is contained in $B(0, r_0)$. The set $\Psi_{\text{rep}}(V)$ is also stable by g , thus g^n is also injective on it. Then, for each n , the map $g^n \circ \Phi_{\text{rep}}^{-1} \circ T_{-n}$ is a composition of injective maps on $T_n(U) \cap V$, and coincides there with Ψ_{rep} . Since the union over n of $T^n(U)$ is the whole complex plane, the claim follows.

Injectivity of h_{nor} on V is similar, since h_{nor} is the union over $n \geq 0$ of the injective maps $T^{-n} \circ \Phi_{\text{attr}}|_{D_{\text{attr}}} \circ g^n \circ \Psi_{\text{rep}}$. \square

Let us introduce a weak notion of convergence of analytic maps: let X, Y be connected Riemann surfaces and let $f_n : U_n \rightarrow Y$ and $f : U \rightarrow Y$ be analytic with U and U_n open subsets of X . Endow Y with any metric compatible with its topology. Let us say that f_n tends to f if for all compact subset K of U , K is eventually contained in U_n and f_n tends to f uniformly on K . This does not depend on the choice of the metric.¹⁶ Note that this does not prevent U_n to have a bigger limit than U . In particular, limits are not unique. We will use the following notation:

$$f_n \rightrightarrows f,$$

which is chosen so to express the fact that f can be contained in limits with a bigger domain. We do not define an associated topology but we will use the notion of sequential continuity with respect to that notion of convergence, as illustrated by the following two claims. For a fixed n , f^n depends continuously on f : if $f_k \rightrightarrows f$ then $f_k^n \rightrightarrows f^n$. Similarly, $f \circ g$ depends continuously on the pair f, g .

Proposition 11 (continuous dependence). *Assume $g_n : U_n \rightarrow \mathbb{C}$ is a sequence of holomorphic maps defined on an open subset U_n of \mathbb{C} containing the origin, with expansion $g_n(z) = z + c_n z^2 + \dots$ at 0, and with $c_n \neq 0$. Assume g is also of this form with $c_g \neq 0$ and that $g_n \rightrightarrows g$. Then $\Phi_{\text{attr}}[g_n] \rightrightarrows \Phi_{\text{attr}}[g]$, $\Psi_{\text{rep}}[g_n] \rightrightarrows \Psi_{\text{rep}}[g]$ and $h_{\text{nor}}[g_n] \rightrightarrows h_{\text{nor}}[g]$.*

Proof. The claim on $h_{\text{nor}} = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$ follows from the claims on Φ_{attr} and Ψ_{rep} .

A compact set K contained in the parabolic basin of g is mapped in $D_{\text{attr}}[g]$ by an iterate g^k . The latter depends continuously on g when k is fixed. Since the center and radius of $D_{\text{attr}}[g]$ depend continuously on g , $g_n^k(K) \subset D_{\text{attr}}[g_n]$ for all n big enough. Continuity, as a function of g , of the restriction of Φ_{attr} to D_{attr} , follows for instance from the third point of Proposition 7 combined with uniqueness of Fatou coordinates: the sequence $\Phi_{\text{attr}}[g_n]$ forms a normal family, and any extracted limit is a Fatou coordinate for g because the functional equation $\Phi_{\text{attr}}[g_n] \circ g_n = T_1 \circ \Phi_{\text{attr}}[g_n]$ passes to the limit, and uniqueness of the normalized Fatou coordinates implies uniqueness of the extracted limit. From the convergence of $\Phi_{\text{attr}}[g_n]$ to $\Phi_{\text{attr}}[g]$ on $D_{\text{attr}}[g]$ we deduce the convergence of $\Phi_{\text{attr}} = \Phi_{\text{attr}}[g_n] \circ g_n^k - k$ to $\Phi_{\text{attr}}[g] \circ g^k - k = \Phi_{\text{attr}}[g]$ on $g^{-k}(D_{\text{attr}}[g])$, and hence on the whole parabolic basin of g .

The proof for Ψ_{rep} is similar. \square

¹⁶This definition has the following equivalent topological formulation. Let $X' = \{0, 1, 1/2, 1/3, 1/4, \dots\} \times X \subset \mathbb{R} \times X$ and embed X' with the topology induced by $\mathbb{R} \times X$. Let $W \subset X'$ be defined by $(0, z) \in W \iff z \in U$ and $(1/n, z) \in W \iff z \in U_n$. Let $F : W \rightarrow Y$ defined by $F(0, z) = f(z)$ and $F(1/n, z) = f_n(z)$. Then $f_n \rightrightarrows f \iff [W \text{ is open relative to } X \text{ and } F \text{ is continuous}]$.

3.5.1. *Transferring to \mathcal{F} .* Fix some $d \in \{2, 3, \dots, \infty\}$ and recall the definition $\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi \in \mathcal{S}\}$. The conclusions of the previous propositions hold for \mathcal{F} . Indeed, the set of restrictions to \mathbb{D} of maps $A \circ f \circ A^{-1}$ with $A(z) = 4z$ satisfies the assumptions of the propositions. First, the set of Schlicht maps \mathcal{S} is compact, and by Koebe's one quarter theorem, the domain of their reciprocal contains $B(0, 1/4)$. The restriction of these reciprocals on $B(0, 1/4)$ forms a compact family. All maps $f \in \mathcal{F}$ have a parabolic fixed point at the origin and the fact that there is only one attracting petal has already been mentioned: f has over $\widehat{\mathbb{C}}$ only three singular values: 0, which is fixed, ∞ which is outside the domain of definition, and a third point; it then follows from Fatou's theorem (Theorem 3) that there can be only one cycle of attracting petals, thus only one petal: $c_f \neq 0$. This is therefore also the case for the conjugate map $A \circ f \circ A^{-1}$. Call \tilde{f} the restriction of $A \circ f \circ A^{-1}$ to \mathbb{D} . The conclusions of the previous propositions are easily transposed from \tilde{f} back to f : for instance, normalized Fatou coordinates satisfy $\Phi_{\text{attr}}[\tilde{f}](z) = \Phi_{\text{attr}}(A(z))$ for all z in the domain of the left hand side (it is contained in the domain of the right hand side but not necessarily equal to it, because \tilde{f} is a restriction).

Recall h_{nor} has the following expansion: $h_{\text{nor}}(z) = z + a_{\text{up/down}} + o(1)$ as $\text{Im}(z) \rightarrow \pm\infty$, where a_{up} and a_{down} are two complex constants. For any map in the class \mathcal{F} , denote $\{0, v_f, \infty\}$ its singular values.¹⁷ The corresponding map h_{nor} has a set of singular values of the form $v_h + \mathbb{Z}$ where

$$v_h = v_h[f] = \Psi_{\text{rep}}[f](v_f).$$

For any $d \in \{2, 3, \dots, \infty\}$, the set \mathcal{F} is sequentially compact, for the notion of convergence defined above. By this we mean that every sequence $f_n \in \mathcal{F}$ has a subsequence f_k such that $f_k \rightrightarrows f$.⁽¹⁸⁾ A sequentially continuous real valued function over a sequentially compact set is bounded. This implies the following proposition.

Proposition 12. *For any $d \in \{2, 3, \dots, \infty\}$ over the class \mathcal{F} , the following holds:*

- (1) *(bound in the normalized attracting Fatou coordinates)*
 $\exists M$ such that $\forall f \in \mathcal{F}$, $|\text{Im}(v_h)| \leq M$.
- (2) *(bound on the horn map at the ends of the cylinder)*
 $\exists M$ such that $\forall f \in \mathcal{F}$, $|a_{\text{up}}[f]| \leq M$ and $|a_{\text{down}}[f]| \leq M$.
- (3) *(bound in the normalized repelling Fatou coordinates)*
 $\exists M$ such that $\forall f \in \mathcal{F}$, the main¹⁹ upper and lower chessboard boxes of h_{nor} respectively contain the half planes $\text{Im}(z) > M$ and $\text{Im}(z) < -M$.

Proof. The map $f \in \mathcal{F} \mapsto v_h \in \mathbb{C}$ is sequentially continuous by Proposition 11. The set \mathcal{F} being sequentially compact, its image by $f \mapsto v_h$ is sequentially compact in \mathbb{C} (i.e. compact) thus bounded. The first point follows.

For the second, by periodicity and the maximum principle and according to the expansion, the distance $|h_{\text{nor}}(z) - z|$ is bounded over $\text{Im}(z) > h+1$ by its supremum over a segment of length 1 inside the line $\text{Im}(z) = h+1$, for instance the segment $[i(h+1), 1+i(h+1)]$. Continuous dependence implies the distance is uniformly bounded as f varies in \mathcal{F} . Since a_{up} is the limit of this difference as $\text{Im}(z) \rightarrow +\infty$, this implies the bound on a_{up} . The proof is similar for a_{down} .

For the third, we will use the following trick: first h_{nor} is an analytic isomorphism commuting with $T_1(z) = z + 1$ from the upper and the lower structural boxes to one of the half plane delimited by $v_h + \mathbb{R}$. By Koebe's one quarter theorem, the

¹⁷It turns out that v_f is independent of f for a fixed d , but we will not use that fact.

¹⁸Note that if we restrict our notion of convergence to \mathcal{F} , we recover uniqueness of the limit.

¹⁹terminology introduced in Section 3.4

upper box must contain $\operatorname{Im}(z) > \frac{\log 4}{2\pi} + \operatorname{Im}(v_h) - \operatorname{Im}(a_{\text{up}})$. The previous bounds allows to conclude. The proof is similar for the other half plane. \square

Let us now prove an independent proposition. Let f be a holomorphic map with a parabolic fixed point with only one attracting petal, whose immediate basin we denote A . Assume that A contains only one singular value v of f (this is the case for $f \in \mathcal{F}$). Then we can apply the universality proposition and we know that $\Phi_{\text{attr}} : A \rightarrow \mathbb{C}$ is structurally equivalent to $\Phi_{\text{attr}}[B_d]$ for some $d \in \{2, 3, \dots, \infty\}$. The singular values of Φ_{attr} are ∞ and the points of the form $\Phi_{\text{attr}}(v) - n$ with $n > 0$ (see for instance Proposition 2 in [BE02], where a notion of *ramified cover* is used: their proposition implies that Φ_{attr} is a cover outside ∞ and the critical values).

Proposition 13. *Under these conditions, the preimage Γ by Φ_{attr} of the horizontal half line $\Phi_{\text{attr}}(v) + [0, +\infty[$ has a connected component \mathcal{C} that is a curve starting from the singular value of f in A and ending at the parabolic point. It is stable: $f(\mathcal{C}) \subset \mathcal{C}$.*

This curve will be called the *principal curve*. It contains in particular the orbit of the singular value of f . Note that all connected components of Γ are curves since the horizontal half line considered contains no singular value of Φ_{attr} .

Proof. One way is to prove the proposition for B_d , which is easy because the latter map is real preserving and its singular value is on the real line, and its explicit formula allows to compute the derivative, etc. . . and then it transfers immediately to f by universality.

Let us give here another proof. Let $v' = \Phi_{\text{attr}}(v)$. Consider the open set $O = \mathbb{C} \setminus]-\infty, v' - 1]$. It is simply connected and does not intersect the set of singular values $v' - \mathbb{N}^*$ of Φ_{attr} . Thus Φ_{attr} is a holomorphic bijection from each component of $\Phi_{\text{attr}}^{-1}(O)$ to O and this applies in particular to the component O_{attr} containing a petal. Now let \mathcal{C} be the preimage by the restriction of Φ_{attr} to O_{attr} of the horizontal half line $v' + [0, +\infty[$. The set \mathcal{C} satisfies the proposition. \square

Let us go back to maps $f \in \mathcal{F}$. As we remarked before, convergence of maps $f_n \rightrightarrows f$ where f_n and f belong to \mathcal{F} is well behaved: limits are unique and in fact it is equivalent to the classical notion of convergence of a sequence with respect to a (metrizable) topology making \mathcal{F} compact: Indeed, let $f_n, f \in \mathcal{F}$. Write $f_n = \mathcal{R}[B_d] \circ \phi_n^{-1}$ and $f = \mathcal{R}[B_d] \circ \phi^{-1}$ with ϕ_n and $\phi \in \mathcal{S}$ (uniquely determined). Then the following are equivalent:

- (1) $f_n \rightrightarrows f$,
- (2) for some $\varepsilon > 0$, the map f_n tends to f uniformly on $B(0, \varepsilon)$,
- (3) for some $\varepsilon > 0$, the map ϕ_n tends to ϕ uniformly on $B(0, \varepsilon)$,
- (4) ϕ_n tends to ϕ uniformly on every compact subsets of \mathbb{D} .

A proof of (3) \implies (4) is for instance given by compactness of \mathcal{S} together with analytic continuation of equalities. The last three notions of convergence are easily metrized and all endow \mathcal{F} with the *same* topology. It is Hausdorff and compact for this topology. The map $\mathcal{S} \rightarrow \mathcal{F}$, $\phi \mapsto \mathcal{R}[B_d] \circ \phi^{-1}$ is a homeomorphism.

Recall that we denote $0, v_f, \infty$ the singular values of f over $\hat{\mathbb{C}}$. It turns out that the class \mathcal{F} has been defined so that v_f does not depend on f , but let us temporarily ignore that.

Lemma 14 (uniform bound on the trapping time). *For any $r > 0$, denote $D_r[f]$ the disk of diameter $[0, re^{i\alpha}]$ where α is the direction of the attracting axis of f . There exists $n_0 \in \mathbb{N}$ such that $\forall f \in \mathcal{F}$, $f^{n_0}(v_f) \in D_r[f]$.*

Proof. Consider $r' = \min(r, r_0)$ where r_0 is provided by Proposition 7. The set $D_{r'}[f]$ is an attracting petal for f and is contained in $D_r[f]$. For each $f \in \mathcal{F}$ it takes a finite number of iterates for v_f to be trapped by $D_{r'}[f]$. The same number of iterates is enough for nearby²⁰ maps in \mathcal{F} . By compactness²¹ of \mathcal{F} , it follows that there is $n_0 \in \mathbb{N}$ such that $\forall f \in \mathcal{F}, \exists n \leq n_0$ such that $f_n(v_f) \in D_{r'}[f]$. Since $D_{r'}[f]$ is a trap this implies $f^{n_0}(z) \in D_{r'}[f]$ and thus $\in D_r[f]$. \square

We will also use a slightly stronger statement:

Lemma 15. *There exists $n_0 \in \mathbb{N}$ and $\eta_0 > 0$ such that $\forall f \in \mathcal{F}, f^{n_0}(B(v_f, \eta_0)) \in D_r[f]$.*

Proof. Done by compactness as above, using the following modification of the local statement, which is immediate by continuity for \rightarrow of $f \mapsto f^n$ for a fixed n : for each $f \in \mathcal{F}$ and each n such that $f^n(v_f) \in D_r[f]$, there is $\eta > 0$ such that for all maps $g \in \mathcal{F}$ close enough to f the n -th iterate of g sends $B(v_g, \eta)$ in $D_r[g]$. \square

We have not checked if all compactness arguments in the rest of the article can be reformulated using \rightarrow only. This is not the main point, however. Moreover, since there is on \mathcal{F} a topology for which convergence of sequences is equivalent to \rightarrow , in the sequel we will use compactness of \mathcal{F} for this topology and convergence of sequences in \mathcal{F} w.r.t. this topology. Recall it is a metrizable topology for which \mathcal{F} is compact.

Below, $d_{\mathbb{C}}$ refers to the Euclidean distance on \mathbb{C} and if U is a open subset of \mathbb{C} whose complement has at least two points, d_U denotes the hyperbolic distance on U . Let $\mathcal{C} = \mathcal{C}[f]$ be the curve introduced in Proposition 13.

Lemma 16. *For $f \in \mathcal{F}$, let $PS(f)$ the orbit of the singular value v_f of f . The following holds:*

- (1) *The sets $\mathcal{C}[f]$ and $\overline{PS}(f)$ depend continuously on f for the Hausdorff topology on compact subsets of \mathbb{C} .*
- (2) $\sup \{|z| \mid z \in PS(f), f \in \mathcal{F}\} < +\infty$
- (3) $\sup \{d_{\text{Dom}(f)}(0, z) \mid z \in PS(f), f \in \mathcal{F}\} < +\infty$
- (4) $\inf \{d_{\mathbb{C}}(z, \partial \text{Dom}(f)) \mid z \in PS(f), f \in \mathcal{F}\} > 0$

Proof. Let us use the same notations as in Lemma 14. For any $r \leq r_0$, denote $D_r = D_r[f]$: it is an attracting petal for f . Let $n_0(r) = n_0$ be provided by Lemma 14. For a fixed $m < n_0(r)$, $f^m(v_f)$ depends continuously on f . The rest of the orbit of v_f is contained in D_r . Continuity of $\overline{PS}(f) = PS(f) \cup \{0\}$ follows, as well as the point 2. For point 4, note that $B(0, 1/4) \subset \text{Dom}(f)$ (this follows from Koebe's 1/4 theorem). Choose now $r = \min(r_0, 1/8)$. For each fixed $m < n_0 = n_0(r)$, the distance from $f^m(v_f)$ to $\partial \text{Dom}(f)$ reaches a positive minimum as f varies in \mathcal{F} , again by continuity and compactness. For $m \geq n_0$, this distance is $\geq 1/8$. For point 3 first note that, on one hand for $m \geq n_0$, $f^m(v) \in B(0, 1/8)$ and thus $d_{\text{Dom}(f)}(0, f^m(v)) \leq 1$ (better constants can be computed but that is not the point here). Let us now use the sets O and O_{attr} introduced in Proposition 13. The map Φ_{attr} is a holomorphic bijection from O_{attr} to $O = \mathbb{C} \setminus]-\infty, v' - 1]$ and the set $X = \{f^m(v) \mid 0 \leq m < n_0\}$ is the preimage by this map of $v' + \{0, 1, \dots, n_0 - 1\}$. Therefore the $\text{Dom}(f)$ hyperbolic distance from X to $B(0, 1/8)$ is \leq the hyperbolic distance in O from v' to $v' + n_0$, which is itself $< n_0$. \square

²⁰We may use the topology on \mathcal{F} , in which case it means that the same iterate is enough for all maps in a neighborhood. Or we may use the notion \rightarrow , in which case it means that for all sequence $f_n \rightarrow f$, this iterate is eventually enough.

²¹cover argument or sequence argument

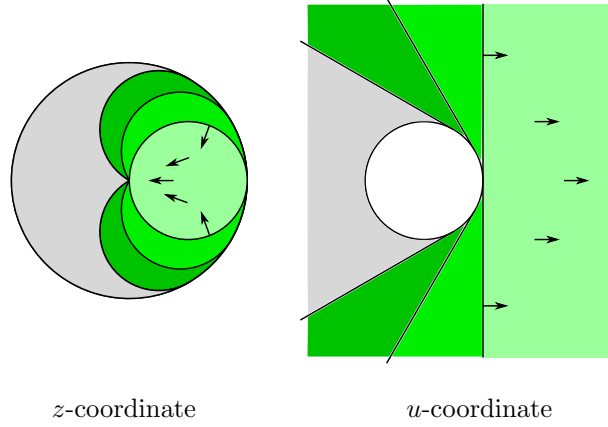


Figure 22: Bigger domains for Fatou coordinates. On the left : z -coordinate and different domains $D_\theta(r_0)$ (in this example the attracting axis is the positive reals), $\theta - 90^\circ = 0, 30^\circ, 60^\circ$, difference between these regions are highlighted in different colors. On the right, the u -coordinate, with $u = -1/c_f z$, and the corresponding domains $W_\theta(R_0)$. In light green are D_{attr} and H_{attr} .

3.5.2. Lemmas for the second step. In the second step of the proof of the main theorem, we will need some control on the variation of Fatou coordinates in terms of the variation of the map. For this we first need to extend Fatou coordinates to bigger domains, as in [Shi00].

Let $\theta \in [\pi/2, \pi]$ and $W_\theta(r)$ denote the following domain: it contains a right half plane and is bounded by the arc of circle of center 0, radius r and argument ranging from $-(\theta - \pi/2)$ to $\theta - \pi/2$, and by the two half lines continuing this arc tangentially to the circle (see Figure 22). For $\theta = \pi/2$ and $r = R_0[g] = 1/|c_g|r_0$, this domain is exactly the half plane, image of D_{attr} in the u -coordinates of g .

Lemma 17. *There is r_0 such that for all $g \in \mathcal{G}$, the change of variable $w = u - \gamma_g \log_p u$ is injective on the non-convex set $W_\pi(R_0[g])$.*

Proof. Let us give a computational but elementary proof of this fact. Write $u = re^{i\theta}$ and $u' = r'e^{i\theta'}$ with $\theta, \theta' \in]-\pi, \pi[$ and note that $r, r' \geq R_0[g] \geq 1/r_0 \inf_{g \in \mathcal{G}} |c_g|$. Then $|r - r'| \leq |u - u'|$ and $|e^{i\theta} - e^{i\theta'}| \leq |u - u'|/\min(r, r')$. If $|\theta - \theta'| \leq \pi$ (case 1) then $|\theta - \theta'| \leq (\pi/2)|e^{i\theta} - e^{i\theta'}|$. Otherwise (case 2), let us just use that $|\theta - \theta'| \leq 2\pi$. Now $w = w'$ means $u - u' = \gamma_g(\log r' - \log r) + \gamma_g i(\theta' - \theta)$ whence (case 1) $|u - u'| \leq \frac{|\gamma_g|(1+\pi/2)}{\min(r, r')}|u - u'|$ therefore $u - u' = 0$ provided r_0 was chosen big enough (independently of g). Or (case 2) $|u - u'| \leq \frac{|\gamma_g|}{\min(r, r')}|u - u'| + 2\pi|\gamma_g|$. In the second case, choose r_0 small enough (independently of g) so that $\frac{|\gamma_g|}{\min(r, r')} \leq 1/2$. Then $|u - u'| \leq 4\pi|\gamma_g|$. Since $\theta - \theta' > \pi$ the points u and u' must have opposite imaginary part and one of them at least has negative real part. Since they belong to $W_\pi(R_0[g])$, which does not contain the half strip of equation “ $\text{Re } z \leq 0$ and $-R_0[g] \leq \text{Im } z \leq R_0[g]$ ”, we get in particular that $|u - u'| > R_0[g]$. So if we choose r_0 small enough so that, $\forall g \in \mathcal{G}$, $R_0[g] > 4\pi|\gamma_g|$, this is impossible. \square

Proposition 18. *Let $\theta \in [\pi/2, \pi[$. Proposition 7 still holds if we replace D_{attr} with the domain $D_\theta(r_0)[g]$ whose image in the u -coordinate is $W_\theta(R_0[g])$ where $R_0[g] = 1/|c_g|r_0$, and if we replace the condition on ζ by $\zeta(x) = -|x \tan(\theta - \pi/2)| + o(x)$. Similar statements hold for repelling Fatou coordinates.*

Proof. The proof carries over with little modification. The constant $1/4$ has to be replaced by a smaller constant (by $\sin \theta$) when θ is too close to π . Injectivity of the change of variable $w = u - \gamma_g \log_p u$ on the non-convex set $W_\theta(R_0)$ follows from the previous lemma. For the uniform bound on $\sum M_3/|u_n|^2$: divide the orbit of u_n into three parts, according to $\operatorname{Re}(u_n)$ being in $] -\infty, -R_0[$, in $[-R_0, R_0]$, or in $]R_0, +\infty[$. In the central part, there is a uniformly bounded number of u_n . The two other parts are bounded exactly like before. \square

Choose any θ with $\pi/2 < \theta < \pi$. Let $\Xi[g](w) = \Phi(z)$, where $u = -1/c_g z$ and $w = u - \gamma_g \log_p(u)$: we take Ξ defined on the image of $W_\theta(R_0[g])$ by $u \mapsto w$. Note that this changes of coordinates depends on g . Choose any other $\theta' < \theta$. Then, by the estimates in Propositions 7 and 18, there exists $R_2 > 0$ such that for all $g \in \mathcal{G}$, the domain and the range of $\Xi[g]$ contains $W_{\theta'}(R_2)$. Recall that maps in \mathcal{G} are assumed to be defined on \mathbb{D} .

Proposition 19. *Let $r' \in]0, 1[$. There exists $M > 0$, $R_1 > R_2$ and $\varepsilon_0 > 0$ such that for all $f, g \in \mathcal{G}$ with $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ then $\forall w \in \mathbb{C}$ with $w \in W_{\theta'}(R_1)$, $|\Xi[f](w) - \Xi[g](w)| \leq M \sup_{B(0, r')} |f - g|$.⁽²²⁾*

Proof. A trick to shorten the proof is to use holomorphic dependence of Fatou coordinates w.r.t. the map. Let $\|f - g\| = \sup_{B(0, r')} |f - g|$. Let $c_0 = \inf |c_g|$ over all $g \in \mathcal{G}$. Let first ε_0 be such that the sum h of a map in \mathcal{G} with a holomorphic map defined on $B(0, r')$ and with a double root at the origin and sup norm $\leq \varepsilon_0$, satisfies $|c_h| > c_0/2$. Let \mathcal{H} be the union of \mathcal{G} and of all the maps of the form $h_t = f + t \frac{\varepsilon_0}{\|f - g\|} (g - f)$ where $t \in \mathbb{D}$, $f, g \in \mathcal{G}$ and $\|f - g\| \leq \varepsilon_0$. Then \mathcal{H} is compact (for the topology associated to uniform convergence on compact subsets of $B(0, r')$) and, conjugating its members by $z \mapsto z/r'$ and restricting to \mathbb{D} , gives a family satisfying the hypotheses of Propositions 7 and 18. Using the latter with any $\theta > \pi/2$ and the remark that follows with $\theta' = \pi/2$, we see that maps $h \in \mathcal{H}$ all have a function $\Xi[h]$ that is defined on a set containing $\operatorname{Re}(w) > R_1$ for some R_1 independent of h . Moreover this function depends holomorphically on $t \in \mathbb{D}$ (recall the definition of Φ as a limit of $w_n - n$ and realize that w_n depends holomorphically on w_n) and its difference with $w \mapsto w$ is uniformly bounded, hence $\Xi[h_t](w) - \Xi[f](w)$ is also bounded. The claim follows by Schwarz's inequality²³ applied to $t \mapsto \Xi[h_t](w) - \Xi[f](w)$. \square

Similarly, Proposition 19 holds word for word with Ξ replaced by Ξ^{-1} , i.e.:

Proposition 20. *Let $r' \in]0, 1[$. There exists $M > 0$, $R_1 > R_2$ and $\varepsilon_0 > 0$ such that for all $f, g \in \mathcal{G}$ with $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ then $\forall \zeta \in \mathbb{C}$ with $\zeta \in W_{\theta'}(R_1)$, $|\Xi^{-1}[f](\zeta) - \Xi^{-1}[g](\zeta)| \leq M \sup_{B(0, r')} |f - g|$.*

Proof. This follows from the above proposition applied to some θ'' between θ and θ' , and from the fact that, the derivative of Ξ_g is uniformly bounded²⁴ over $g \in \mathcal{G}$. Computations are left to the reader. \square

Remark. Note that since the maps Ξ_g and Ξ_g^{-1} all differ from identity by a bounded amount that is independent of $g \in \mathcal{G}$, it follows that in both propositions, by increasing the value of M , we can remove the assumption $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$.

From Proposition 19, we deduce the following control, which is somewhat weaker:

²²A better bound holds, that decays when w tends to infinity, but it will not be used here.

²³We mean: if $f : \mathbb{D} \rightarrow \mathbb{C}$ is holomorphic and satisfies $f(0) = 0$ and $\sup |f| < +\infty$ then $|f(z)| \leq |z| \sup |f|$.

²⁴increase R_2 by 1 and use Cauchy's formula and the uniform bound on $\Xi - \operatorname{id}$

Proposition 21 (variation of Fatou coordinates). *Let $r' \in]0, 1[$. Let R_1 be given by Proposition 19. Let $\theta'' < \theta'$. Then there exists $M > 0$, $R_3 > R_1$ and $\varepsilon_0 > 0$ such that for all $f, g \in \mathcal{G}$ with $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ and $\forall z \in \mathbb{C}$ with $-1/c_f z \in W_{\theta''}(R_3)$, then $-1/c_g z \in W_{\theta'}(R_1)$ and $|\Phi_{\text{attr}}[f](z) - \Phi_{\text{attr}}[g](z)| \leq M \sup_{B(0, r')} |f - g|/|z|$. The same holds for repelling Fatou coordinates.*

Proof. Let $d = \sup_{B(0, r')} |f - g|$. The claim $-1/c_g z \in W_{\theta'}(R_1)$ follows from continuity of $g \mapsto c_g$ and its non-vanishing: given any $R_3 > 1$ and $\theta'' < \theta'$, a small enough d will ensure that the quotient c_g/c_f is close enough to 1 so that an element of $W_{\theta''}(R_3)$ multiplied by c_f/c_g is still in $W_{\theta'}(R_1)$. Now $\Phi_{\text{attr}}[f](z) = \Xi[f](w_1)$ and $\Phi_{\text{attr}}[g](z) = \Xi[g](w_2)$ with $w_1 = u_1 - \gamma[f] \log_p(u_1)$ and $w_2 = u_2 - \gamma[g] \log_p(u_2)$ with $u_1 = -1/c_f z$ and $u_2 = -1/c_g z$. The constants c , $1/c$ and γ are Lipschitz functions of $f \in \mathcal{G}$ w.r.t. the distance d . Now, under the assumption $d \leq \varepsilon_0$, we successively get $|u_1 - u_2| \leq M_1 d/|z|$, $|w_1 - w_2| \leq M_2 d/|z|$ (because $|\log_p u| \ll |u|$), then we decompose $|\Xi[f](w_1) - \Xi[g](w_2)| \leq |\Xi[f](w_1) - \Xi[g](w_1)| + |\Xi[g](w_1) - \Xi[g](w_2)|$. The first term is dealt with using Proposition 19 and the second term using the fact that there is a uniform bound on Ξ' . \square

Remark.

- Here, the condition $\sup_{B(0, r')} |f - g| \leq \varepsilon_0$ cannot be removed.
- Also, in the conclusion $|\Phi_{\text{attr}}[f](z) - \Phi_{\text{attr}}[g](z)| \leq M \sup_{B(0, r')} |f - g|/|z|$, the factor $1/|z|$ cannot be removed because $\Phi_{\text{attr}}[f](z) \sim -1/c_f z$ and c_f varies with f .

Let us stress again that, though maps in \mathcal{F} are not defined on the unit disk, they are all defined in $B(0, 1/4)$ and the results above easily transfer to \mathcal{F} by a homothety. (See Section 3.5.1.)

3.6. Step 1: contraction argument (i.e. there is a lot of room). Fix $d \in \mathbb{N}$ with $2 \leq d < \infty$: we now exclude $d = +\infty$. In this section we will define constants c_1, c_2, \dots . They all depend on d but not on $f \in \mathcal{F}$.

Recall that $\mathcal{R}[B_d]$ is defined on the unit disk and has derivative one at the origin. Recall the definition of the set \mathcal{S} of Schlicht maps: univalent holomorphic maps $\phi : \mathbb{D} \rightarrow \mathbb{C}$ such that $\phi(z) = z + \mathcal{O}(z^2)$. Recall that $\mathcal{F} = \{\mathcal{R}[B_d] \circ \phi^{-1} \mid \phi \in \mathcal{S}\}$, and that for all $f \in \mathcal{F}$, the map $\mathcal{R}[f]$ is again in \mathcal{F} , for an appropriate normalization. Since all maps in \mathcal{F} have the same unique critical value, this normalization coincides with the one numbered 3 on page 8, which we called “by the critical value”.

Let $f \in \mathcal{F}$:

$$f = \mathcal{R}[B_d] \circ \phi_1^{-1}$$

where $\phi_1 \in \mathcal{S}$. Denote

$$U_1 = \phi_1(\mathbb{D}) = \text{Dom}(f).$$

Let $L(\varepsilon)$ be the hyperbolic radius of $B(0, 1 - \varepsilon)$ in \mathbb{D} :

$$L(\varepsilon) = \tanh^{-1}(1 - \varepsilon) = \frac{1}{2} \log \frac{2 - \varepsilon}{\varepsilon}.$$

In particular

$$\frac{1}{2} \log \frac{1}{\varepsilon} \leq L(\varepsilon) \leq \frac{\log 2}{2} + \frac{1}{2} \log \frac{1}{\varepsilon}.$$

Since $\mathcal{R}[f]$ belongs to \mathcal{F} (Theorem 1), there exists $\phi_2 \in \mathcal{S}$ such that:

$$\mathcal{R}[f] = \mathcal{R}[B_d] \circ \phi_2^{-1}.$$

The map ϕ_2 is an isomorphism from \mathbb{D} to the domain of definition of $\mathcal{R}[f]$.

Denote by $A \subset U_1$ the immediate basin of the parabolic fixed point 0 of f . Let U_u denote the connected component of $\text{Dom}(h_{\text{nor}})$ that contains an upper half

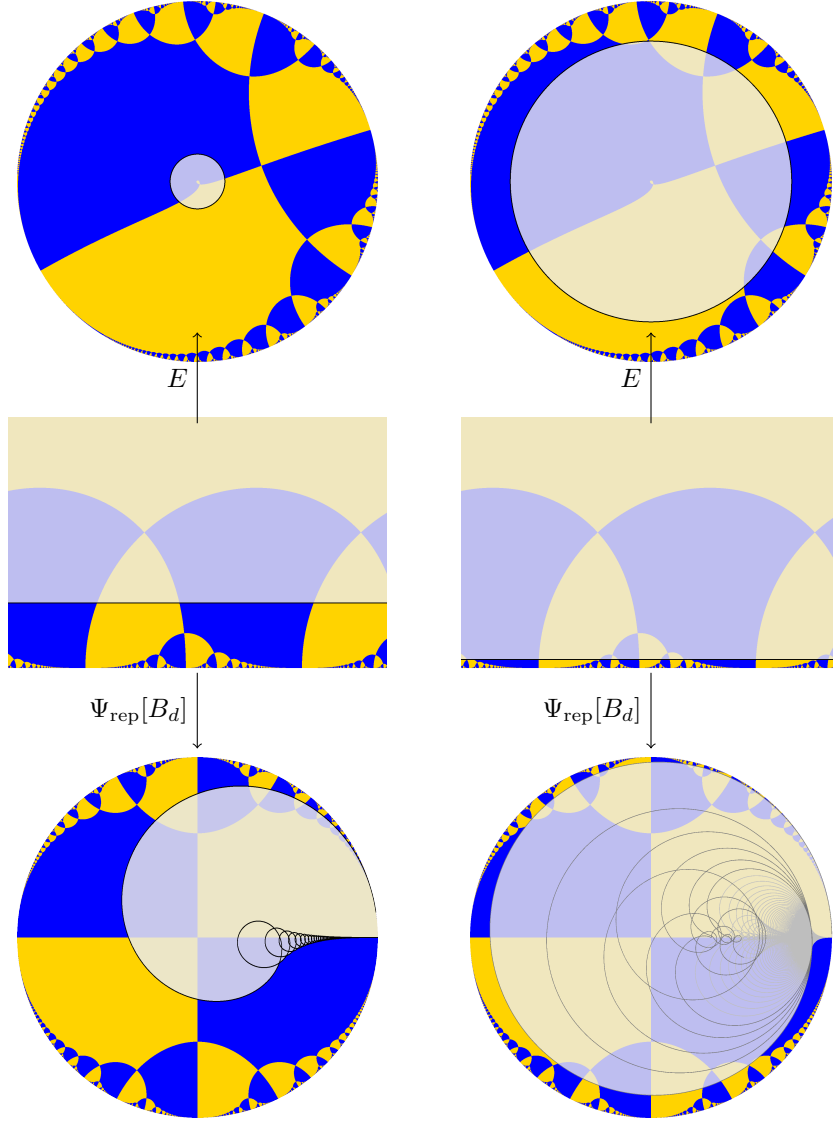


Figure 23: The set $C[B_d]$ (light tones) for $d = 2$ and (left) $\varepsilon \approx 0.85$ (this is quite high a value for an ε) and (right) $\varepsilon \approx 0.22$.

plane. It is also equal to the connected component of $\Psi_{\text{rep}}^{-1}(A)$ that contains an upper half plane. Denote by $C \subset A$ the following set, which is the object under study in the present section:

$$C = C[f] = \Psi_{\text{rep}}\left(U_u \Vdash (1 - \varepsilon)\right)$$

(The notation \Vdash has been introduced in Section 3.1). We claim it can be rewritten as

$$C = \phi_3(C[B_d])$$

where $\phi_3 : \mathbb{D} \rightarrow A$ is the conformal isomorphism conjugating B_d to $f|_A$. Indeed, according to the complement after Theorem 2, $\phi_3 \circ \Psi_{\text{rep}}[B_d] = \Psi_{\text{rep}}[f] \circ \phi_4$ for some conformal map ϕ_4 from $\mathbb{H} = U_u[B_d]$ to $U_u[f]$, commuting with T_1 , thus $\phi_4(U_u[B_d] \Vdash (1 - \varepsilon)) = U_u[f] \Vdash (1 - \varepsilon)$.

The set $C[B_d]$, which is equal to $\Psi_{\text{rep}}[B_d](H(\varepsilon))$ where $H(\varepsilon) = E^{-1}(B(0, 1 - \varepsilon))$ is the half plane defined by “ $\text{Im } z > \frac{1}{2\pi} \log \left(\frac{1}{1-\varepsilon} \right)$ ”, depends only on d and ε , not on f , and is forward invariant under B_d . Figure 23 shows examples of sets $C[B_d]$.

In this section we will prove:

Proposition 22. *There exists c, c' and $\xi > 0$ (these constants depend on d) such that for all $\varepsilon < \xi$, there exists $\varepsilon' > 0$ satisfying*

$$\log \frac{1}{\varepsilon'} \leq c' + c \log \left(1 + \log \frac{1}{\varepsilon} \right)$$

such that for all $f \in \mathcal{F}$,

$$C \subset \text{Dom}(f) \odot (1 - \varepsilon').$$

The notation $U \odot r$ has been introduced in Section 3.1.

We begin with an easy lemma:

Lemma 23. *The set $C[B_d]$ is contained within hyperbolic \mathbb{D} -distance $\leq c_2 + L(\varepsilon)$ of the upper main chessboard box of B_d .*

Proof. The upper chessboard box of B_d is the image by $\Psi_{\text{rep}}[B_d]$ of an open set that contains a half plane “ $\text{Im}(z) > M_d$ ” and is contained in another half plane strictly smaller than \mathbb{H} . Recall that $C[B_d] = \Psi_{\text{rep}}[B_d](H(\varepsilon))$ with $H(\varepsilon) = \{z \in \mathbb{H} : \text{Im}(z) > \frac{1}{2\pi} \log \left(\frac{1}{1-\varepsilon} \right)\}$. For ε big, $H(\varepsilon) \subset \{z \in \mathbb{H} : \text{Im}(z) > M_d\}$. For other values of ε , every point in $H(\varepsilon)$ can be joined to “ $\text{Im}(z) > M_d$ ” by a vertical segment of hyperbolic length in \mathbb{H} at most $\frac{1}{2} \left(\log M_d - \log \frac{\log \frac{1}{1-\varepsilon}}{2\pi} \right)$. Since $\Psi_{\text{rep}}[B_d] : \mathbb{H} \rightarrow \mathbb{D}$ contracts hyperbolic metrics and $\log \frac{1}{\log \frac{1}{1-\varepsilon}} \leq \log \frac{1}{\varepsilon} \leq 2L(\varepsilon)$, the lemma follows. \square

Note that $\phi_3 : \mathbb{D} \rightarrow A$ is an isometry for the respective hyperbolic metrics, and that the upper main chessboard box of B_d is mapped by ϕ_3 to the main upper dynamical chessboard box of A , call it Mudba:

$$\text{Mudba} = \phi_3(\text{Mudba}[B_d]).$$

See Figure 24. From the lemma above, it follows that the set $C = C[f]$ under study is contained within A -hyperbolic distance $c_2 + L(\varepsilon)$ of Mudba. In order to prove an estimate concerning the latter set, we first need the following easy consequence of the compactness of \mathcal{F} :

Lemma 24. *For all $M > 0$ there exists $c' > 0$ such that for all $f \in \mathcal{F}$, the upper main and the lower main chessboard boxes of h_{nor} are both at hyperbolic $\text{Dom}(h_{\text{nor}})$ -distance $\leq c'$ from respectively the half planes $\text{Im}(z) > M$ and $\text{Im}(z) < -M$ (intersected with $\text{Dom}(h_{\text{nor}})$ if necessary).*

Proof. The extended normalized horn map of B_d is defined on $\mathbb{C} \setminus \mathbb{R}$. The upper/lower main chessboard boxes of $h_{\text{nor}}[B_d]$ are at positive Euclidean distance from \mathbb{R} . Recall (see Section 3.3) that we have the following: $h_{\text{nor}}[f] \circ \phi = T_{w[f]} \circ h_{\text{nor}}[B_d]$ where $w[f] = \Phi_{\text{attr}}[f](v_f) - \Phi_{\text{attr}}[B_d](v_{B_d})$ and ϕ is an isomorphism commuting with T_1 from \mathbb{H} , which is the upper connected component of $\text{Dom}(h_{\text{nor}}[B_d])$, to the upper connected component of $\text{Dom}(h_{\text{nor}}[f])$. Therefore, it is enough to prove that $\phi^{-1}(\{z \in \mathbb{H} : \text{Im}(z) > M\})$ contains an upper half plane independent of f , and a similar statement for the lower part. Let us write, as $\text{Im}(z) \rightarrow +\infty$:

$$\phi(z) = z + \tau_f + o(1).$$

From the first point of Proposition 12 it follows that $|\text{Im}(w[f])|$ is bounded over \mathcal{F} . From this and the second point, it follows that $|\text{Im}(\tau_f)|$ is bounded over \mathcal{F} . Now one of Koebe’s inequalities states that $\forall f \in \mathcal{S}$, $|f(z)| \leq \frac{|z|}{(1-|z|)^2}$. Equivalently, $\forall r \in$

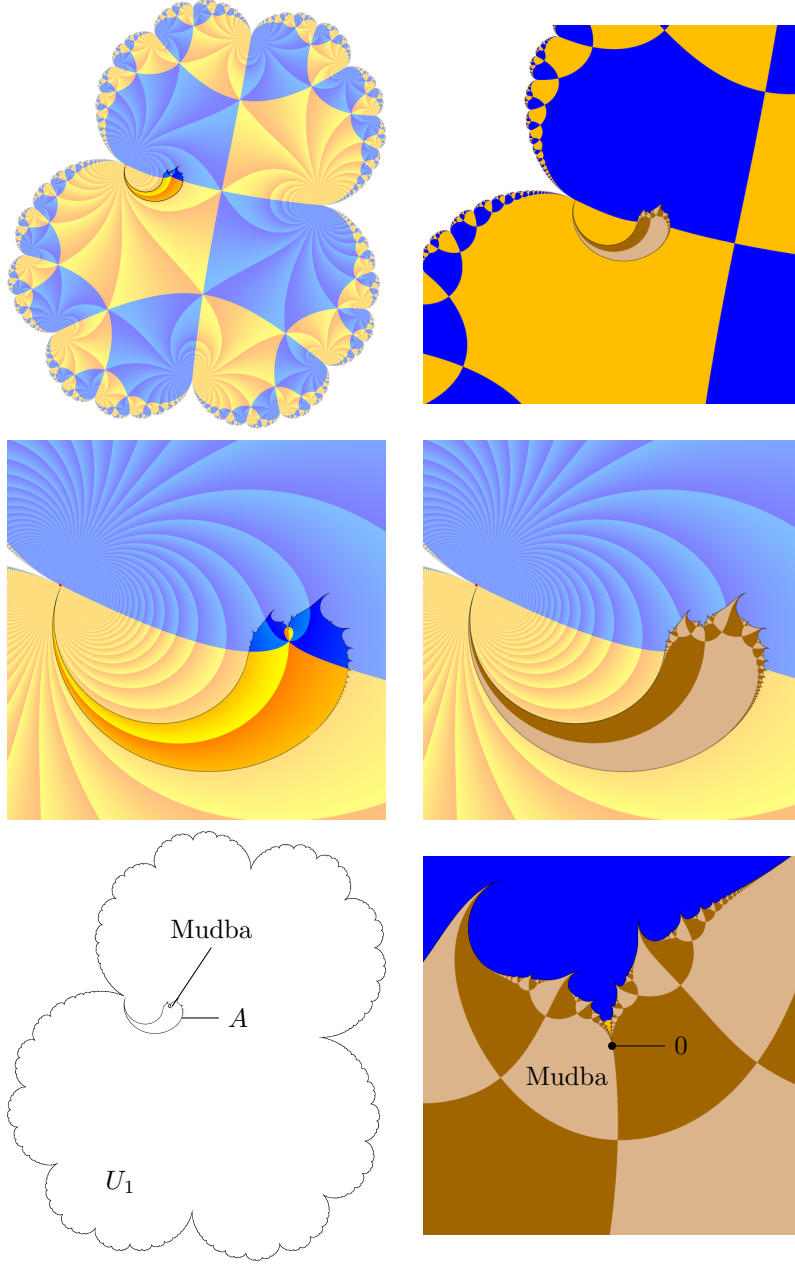


Figure 24: Some open sets associated to $\mathcal{R}[P]$ with $P : z \mapsto z + z^2$: its domain U_1 , its parabolic immediate basin A , and the latter's main upper dynamical box Mudba. The rightmost column features the dynamical chessboard of A in shades of brown. The blue and yellow shades depict the structural chessboard of U_1 .

$]0, 1[, f^{-1}(B(0, r/(1-r)^2)) \supset B(0, r)$. The map $T_{-\tau_f} \circ \phi$ is semi-conjugate by E to a Schlicht map thus: $\phi^{-1}(\text{"Im}(z) > M\text{"})$ contains the half plane $\text{"Im}(z) > M'\text{"}$ where $M' = M'[f] > 0$ is related to $M \in \mathbb{R}$ by $e^{2\pi(M+\text{Im } \tau_f)} = e^{2\pi M'} + e^{-2\pi M'} - 2 = 2(\cosh(2\pi M') - 1)$. Since τ_f is bounded, the constant $M'[f]$ is bounded too. The proof for the lower box is similar. \square

Lemma 25. *Mudba is contained in a hyperbolic U_1 -ball of uniform diameter c_7 .*

Proof. Choose r small enough so that $B(0, 2r) \subset U_1$ for all $f \in \mathcal{F}$. By Proposition 10, there is some $h > 0$ such that for all $f \in \mathcal{F}$, the half planes $\text{Im}(z) > h$ and $\text{Im}(z) < -h$ are mapped by $\Psi_{\text{rep}}[f]$ inside $B(0, r)$. From Lemma 24 the upper box is at distance $\leq c_7$ from “ $\text{Im}(z) > h$ ” for the hyperbolic metric of $\text{Dom}(h_{\text{nor}})$. The map $\Psi_{\text{rep}} : \text{Dom}(h_{\text{nor}}) \rightarrow A$ is holomorphic thus a contraction for hyperbolic metrics, thus the image by Ψ_{rep} of the upper chessboard box is at bounded A -hyperbolic distance of $B(0, r)$ (the latter is not contained in A but it does not matter) and thus at U_1 -hyperbolic distance even smaller, since the inclusion of A in U_1 is a contraction too. \square

By Lemmas 23 and Lemma 25, to fulfill the objectives of Step 1, it is enough to prove that a path of A -hyperbolic length $\leq c_2 + L(\varepsilon)$ starting from Mudba has a U_1 -hyperbolic length much smaller than $c_2 + L(\varepsilon)$. The precise bound obtained will yield Proposition 22. Note that we will in fact bound the U_1^* -hyperbolic length, which is bigger than the U_1 -hyperbolic length, where

$$U_1^* = U_1 \setminus \{0\}.$$

Let us make the following change of coordinates: $w = \log(z)/2i\pi$. Let \tilde{A} be a lift of A : it is a connected and simply connected subset of \mathbb{C} that does not intersect its translates $\tilde{A} + k$ when $k \in \mathbb{Z}$ is non-zero. As a consequence, each horizontal intersects this open set along a union of open segments of length at most 1 (in fact the sum of lengths is at most 1). Thus the distance from any $z \in \tilde{A}$ to the boundary of \tilde{A} is $\leq 1/2$. This implies by Koebe’s 1/4 Theorem:

$$\rho_{\tilde{A}}(z) \geq 1/2$$

(a better bound holds but we do not need it; recall $\rho_U(z)|dz|$ designates the infinitesimal element of hyperbolic metrics on U).

Remark. The set \tilde{A} is unbounded upwards, since the image in \tilde{A} of an attracting petal in A is an infinite finger-shaped domain extending upwards. See Figure 25 for examples. Recall that $f \in \mathcal{F}$ is characterized by the choice of its domain U_1 , which can be any simply connected domain containing the origin with conformal radius 1 w.r.t. the origin. For well chosen unbounded U_1 , the set \tilde{A} is unbounded downwards. One could object that since in the applications, the renormalization operator is iterated, we could restrict to maps in $\mathcal{R}[\mathcal{F}]$ instead of \mathcal{F} , and that maps in $\mathcal{R}[\mathcal{F}]$ all have a uniformly bounded domain of definition, as follows for instance from Proposition 12. But worse happens: even for bounded U_1 , provided its boundary swirls infinitely many times around 0, carefully chosen U_1 will yield a set \tilde{A} whose projection on the real line is unbounded. The latter case is not just a curiosity but does happen for $f = \mathcal{R}[z \mapsto ze^z]$, i.e. the first renormalization of the map $g(z) = ze^z$ which has a parabolic point at the origin, and whose set of singular values are the two asymptotic values ∞ , 0 and the image $g(-1)$ of the unique critical point -1 . Its immediate basin must contain a singular value, and the only possible one is $g(-1)$. Hence the map g satisfies the hypotheses of Theorem 1, thus $f = \mathcal{R}[g] \in \mathcal{F}$. A careful study shows that the domain of definition of $\mathcal{R}[g]$ swirls like above, more precisely that its lifted immediate basin \tilde{A} has infinitely many accesses to infinity by curves asymptotic to some common horizontal line. The map g does not belong to \mathcal{F} but we believe that for all $n > 0$, $\mathcal{R}^n[g]$, that belongs to \mathcal{F} , will have a set \tilde{A} with the same properties. To prove this, one may try and see if there is invariance by \mathcal{R} of the following property for $f \in \mathcal{F}$: let c be the main critical point of f (the one on the boundary of the main upper structural box); let \tilde{f} be a lift of f and let γ be the lift by \tilde{f} starting from c , of the horizontal

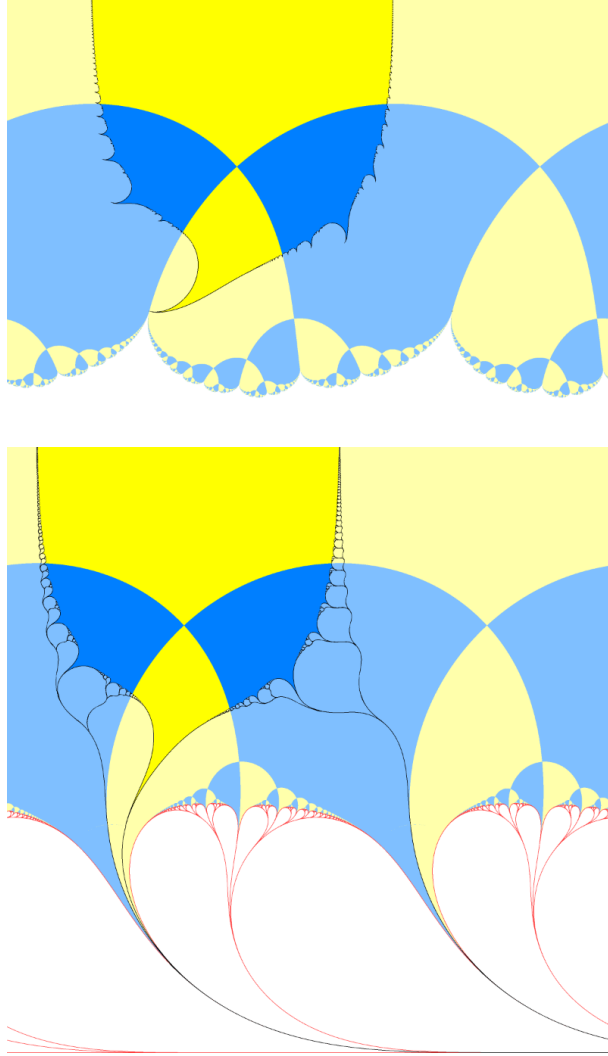


Figure 25: Two examples of lifted immediate parabolic basins \tilde{A} for maps $f \in \mathcal{F}$. Left: $f = \mathcal{R}(z \mapsto z + z^2)$, Right: $f = \mathcal{R}(z \mapsto ze^z)$.

half line $\tilde{f}(c) + [0, +\infty[$, and that intersects the boundary of the upper box only at c ; then $\text{Re}(\gamma)$ tends to infinity.

Now consider a point $z_0 \in C = \phi_3(C[B_d])$ and consider a path γ of A -length at most $c_2 + L(\varepsilon)$ from Mudba to z_0 . Let us apply f once. Then A is mapped to itself and so are C and Mudba. The path γ is mapped to a path $f(\gamma)$ contained in A , from Mudba to $z_1 = f(z_0)$, and by the Schwarz-Pick inequality, the A hyperbolic length of $f(\gamma)$ is \leq that of γ . Consider a lift γ_2 of $f \circ \gamma$ by E (the path $f(\gamma)$ is contained in A , thus does not meet the origin). The Euclidean length of γ_2 is equal to

$$(2) \quad \int_{\gamma_2} |dz| = \int_{\gamma_2} \frac{\rho_{\tilde{A}}(z)|dz|}{\rho_A(z)} \leq 2 \int_{\gamma_2} \rho_{\tilde{A}}(z)|dz| \leq 2(c_2 + L(\varepsilon)).$$

Let us now relate the element of length $\rho_{U_1^*}(z)|dz|$ to $|d \log f(z)/2\pi|$. Let \tilde{f} be the continuous lift of f by E that fixes \tilde{A} : $\tilde{f}: \tilde{U}_1 \stackrel{\text{def}}{=} E^{-1}(U_1) \rightarrow \mathbb{C}$ and $E \circ \tilde{f} = f \circ E$.

The inverse of E is the multivalued function $E^{-1}(z) = \frac{1}{2\pi i} \log z$. Let $\tilde{v} + \mathbb{Z}$ be the set of critical values of \tilde{f} . The map \tilde{f} has no asymptotic value over \mathbb{C} . Denote by \mathbb{C}^\pm the upper half plane and the lower half plane delimited by the horizontal line through these critical values. For all point z mapped to \mathbb{C}^\pm by any branch of $\frac{1}{2\pi i} \log f$, the latter map has inverse branches defined in \mathbb{C}^\pm , with image the f -structural chessboard box containing z . This inverse branch is univalent, except for z in the little loop around 0 where it is infinite-to one. In all cases, these inverse branches map in U_1^* and are non-expanding for the respective hyperbolic metrics as follows:

$$(3) \quad \rho_{U_1^*}(z)|dz| \leq \rho_{\mathbb{C}^\pm}(\zeta)|d \log f(z)/2\pi|$$

where ζ is the image of z by the considered branch of $\frac{1}{2\pi i} \log f$.

Near the boundary of \mathbb{C}^\pm , better estimates hold. For instance:

Lemma 26. *There exists $c_3 > 0$ such that for all $f \in \mathcal{F}$, the following holds. Let \tilde{v} be a critical value of \tilde{f} and V be any connected component of the pre-image by \tilde{f} of the square $\tilde{v} + I + iI$ where $I = [-1/2, 1/2]$. Then the hyperbolic diameter in U_1^* of $E(V)$ is $\leq c_3$.*

Proof. Recall the critical values of \tilde{f} , are the elements of $\tilde{v} + \mathbb{Z}$ and that its only asymptotic value over $\hat{\mathbb{C}}$ is ∞ . Consider the disk $\tilde{v} + \mathbb{D}$ and the component U of \tilde{f}^{-1} that contains V . Then \tilde{f} factors on U as $a \circ \text{pow} \circ b$ where $\text{pow} : \mathbb{D} \rightarrow \mathbb{D}$ is either the identity or the map $z \mapsto z^d$, where $a(z) = \tilde{v} + z$ and where b is an isomorphism from U to \mathbb{D} . Then $a^{-1}(V) = I + iI \subset B(0, 1/\sqrt{2})$ thus $(a \circ \text{pow})^{-1}(V)$ is contained in the Euclidean ball $B(0, \left(\frac{1}{\sqrt{2}}\right)^{1/d})$. The map $b^{-1} : \mathbb{D} \rightarrow E^{-1}(U_1)$ is non-expanding for the respective hyperbolic metrics, and $E : E^{-1}(U_1) \rightarrow U_1^*$ also is, thus the lemma holds with $c_3 =$ the hyperbolic distance in \mathbb{D} from 0 to the d -th root of $1/\sqrt{2}$. \square

Another easy lemma:

Lemma 27. *Let a, b be two points in the hyperbolic plane \mathbb{H} :*

$$\text{Im}(a) \geq \frac{1}{2} \text{ and } \text{Im}(b) \geq \frac{1}{2} \implies d_{\mathbb{H}}(a, b) \leq \log(1 + 2|a - b|).$$

Proof. Use the following formula for the hyperbolic distance in \mathbb{H} :

$$d_{\mathbb{H}}(a, b) = \text{argsh} \frac{|b - a|}{2\sqrt{\text{Im } a \text{ Im } b}},$$

and the inequality $\text{argsh } t \leq \log(1 + 2t)$. \square

So for instance, the hyperbolic distance from i to $i + x$ is a $\mathcal{O}(\log x)$ when $x \rightarrow +\infty$, thus much smaller than x . Recall that the geodesic between a and b is an arc of circle. For the hyperbolic metric, this arc turns out to be much shorter than the straight euclidean line.

Let β_0 be the structural U_1 chessboard box that is a punctured neighborhood of the origin. Recall that we denote $U_1^* = U_1 \setminus \{0\}$. Consider any structural U_1 chessboard box β . Let us call *cubox* the set $\bar{\beta} \cap U_1^*$. Let us endow $U_1^* \setminus f^{-1}(v)$ with the infinitesimal metric induced by pulling back the Euclidean metric by $\frac{1}{2\pi i} \log f$. We call this the *flat metric*. It has a regular and locally flat extension to a neighborhood of the non-critical preimages of v and is singular precisely at the critical preimages of v , where it has a conical point of angle $2\pi d$. Let us call *box-Euclidean distance* the distance induced on U_1^* by this flat metric. Recall that if $\beta \neq \beta_0$, then $\frac{1}{2i\pi} \log f$ is well defined on β and maps it to a half plane \mathbb{C}^\pm . It also maps the cubox $\bar{\beta} \cap U_1^*$ to the closure of this half plane.

In the sequel, we call b_* the cubox that contains a punctured neighborhood of the origin: $b_* = \overline{\beta_0} \cap U_1^*$.

Corollary 28. *Consider two points in a structural U_1 chessboard box b , the distance d_e between these two points for the box-Euclidean distance on b and the distance d_h between these two points for hyperbolic metric on U_1^* . Then*

$$d_h \leq c'_5 + \log(1 + c_5 d_e).$$

Proof. Let us apply $\frac{1}{2\pi i} \log f$ so as to work in a half plane, and to fix ideas, let us assume it is the half plane \mathbb{C}^+ . If any of the two points is at distance $\leq 1/2$ from the boundary then move it up so that it is at distance $1/2$: we get a new pair of points in \mathbb{C}^+ that corresponds to a new pair of points in b . By Lemma 26, each new point is at U_1^* -hyperbolic distance $\leq c_3$ from the former so the U_1^* -hyperbolic distance between the the points in the pair has changed by at most c_3 , and by at most $2c_3$ if we needed to move both points. Similarly, the Euclidean distance between the points in \mathbb{C}^+ has changed by at most 1. By Equation (3) the U_1^* -hyperbolic distance between the two (possibly) new points will be at most their \mathbb{C}^+ -hyperbolic distance. Using Lemma 27, on the latter we get $d_h \leq 2c_3 + \log(1 + 2(d_e + 1)) = (2c_3 + \log 3) + \log(1 + \frac{2}{3}d_e)$. \square

Let \tilde{f} -cuboxes be defined similarly: these are sets of the form $\tilde{b} \cap \tilde{U}_1$ where b is a structural chessboard box of \tilde{f} . The map \tilde{f} is a bijection from such a set to the closed upper or lower half plane. We can endow \tilde{U}_1 with an infinitesimal box-Euclidean metric, by pulling-back by \tilde{f} the canonical Euclidean metric element $|dz|$ on the complex plane. Recall that $f \circ E = E \circ \tilde{f}$, thus we get the following compatibility statements. The projection by E of a cubox is a \tilde{f} -cubox.²⁵ The box-Euclidean metric element on \tilde{U}_1 is the pull-back by E of the box-Euclidean metric element on U_1^* .

The following result is not used here, but we find it interesting:

Lemma 29. *A connected union of cuboxes that includes b_* is simply connected if we add $\{0\}$ to the union.*

Proof. Remove the loop from the parabolic structural chessboard graph of U_1 . Then we get a tree (an infinite tree), on which the union retracts to a connected subset, which is thus simply connected and homotopically equivalent to the union. \square

Note that there are paths in U_1 reaching the boundary, and whose compact subsets are of hyperbolic diameter comparable to their box euclidean length: see Figure 26. An important task is thus to formulate and prove a combinatorial statement (Lemma 31) about the cuboxes that the immediate basin A may cross, and that prevents this kind of behaviour.

Define a chain of boxes to be a finite sequence b_0, b_1, \dots, b_n of cuboxes such that two consecutive elements have non empty intersection, i.e. consecutive boxes are equal or share a side or a corner within U_1 . The integer n is called the length of the chain (with our convention there are $n + 1$ cuboxes in a chain of length n). Define the *combinatorial distance* between cuboxes as the minimal length of chains from one to the other.

Lemma 30. *Let b, b' be cuboxes and consider points $x \in b$ and $x' \in b'$. Then the box-Euclidean distance L between x and x' and the combinatorial distance n between b and b' satisfy:*

$$n \leq \lfloor L \rfloor + 1.$$

²⁵The connected components of the preimage of a cubox by \tilde{f} are \tilde{f} -cuboxes with one notable exception where we get a chain.

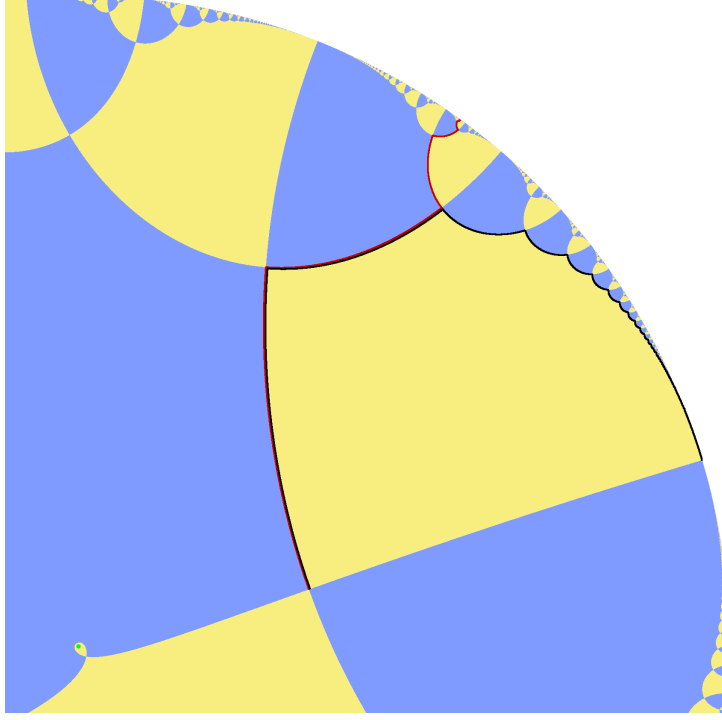


Figure 26: A slow path in black, a quick path in red. The first one stays on the boundary of a single cubox. The other one turns alternately left and right at every corner. Here speed is to be understood as the order of magnitude of the hyperbolic distance from the origin, when the curve is followed at constant box-Euclidean speed (on this picture, it takes the same time to get from a corner to the next one): in the first case it is logarithmic, in the second case linear.

Proof. First case: $L < 1$. Recall that the set of critical values of \tilde{f} is of the form $\tilde{v} + \mathbb{Z}$ for some \tilde{v} and that \tilde{f} has no asymptotic value over \mathbb{C} . Consider a path γ from x to x' and of box-Euclidean length < 1 . Let $\tilde{\gamma}$ be a lift of γ by E . The image of $\tilde{\gamma}$ by \tilde{f} has Euclidean length < 1 in the plane. There will therefore exist $k \in \mathbb{Z}$ such that $\tilde{\gamma}$ is completely contained in the plane minus the translate of $] -\infty, -1] \cup [1, +\infty[$ by $\tilde{v} + k$. The connected components of the pre-image by \tilde{f} of such a slit plane are contained in unions of 2 or $2d$ of \tilde{f} -cuboxes that touch at a common point: this is because there is at most one critical value of \tilde{f} in the slit plane. Now $\tilde{\gamma}$ is contained in such a component hence $n \leq 1$.

In the general case, there is a shortest path from x to x' by Lemma 32 but we can do here without that information: consider a path γ of length close enough to L so as to have the same integer part as L . Let $\varepsilon > 0$ and cut the path into pieces of length $1 - \varepsilon$, except maybe for the last piece for which we require length $\leq 1 - \varepsilon$. Let k be the number of pieces obtained: if ε small enough, $k = \lfloor L \rfloor + 1$. Let x_0, \dots, x_k denote the sequence of starting and end points of these pieces. Let $b_0 = b$, $b_k = b'$ and for $0 < n < k$ let b_n be a cubox containing x_n . From the first case we get that the combinatorial distance between b_n and b_{n+1} is ≤ 1 for $0 \leq n < k$. The combinatorial distance between b and b' is thus $\leq k$. \square

For $n \geq 0$, consider the set \mathcal{B}_n of cuboxes at combinatorial distance $\leq n$ of the cubox b_* . Note that for $n \geq 2$, the set \mathcal{B}_n is a infinite union of cuboxes. The next lemma is illustrated by Figure 27.

Lemma 31. *There exists $c_4 \in \mathbb{N}$ such that $\forall f \in \mathcal{F}$, $A \subset \mathcal{B}_{c_4}$.*

Proof. Let us consider the principal curve $\mathcal{C} = \mathcal{C}[f]$ defined in Proposition 13, starting from $v = v_f$ and ending at 0. Let us use Proposition 7, that provides a disk $D_{\text{attr}} = D_{\text{attr}}[f]$ of uniform diameter r_0 contained in the basin of f and which eventually traps any orbit in the basin. By Lemma 14, the number of iterates needed for the critical value v to enter D_{attr} is bounded over \mathcal{F} : $\exists n_0 \geq 0$ such that $\forall f \in \mathcal{F}$, $f^{n_0}(v) \in D_{\text{attr}}[f]$. The second and third points of Proposition 7 imply that there is a full preimage \mathcal{C}' contained in D_{attr} of the horizontal half line $\Phi_{\text{attr}}(v) + [n_0, +\infty[$ by the Fatou coordinate Φ_{attr} , and that starts from $f^{n_0}(v)$ and ends at the origin. This preimage \mathcal{C}' is a subset of the principal curve \mathcal{C} . (Indeed \mathcal{C} has points in common with D_{attr} , because $\mathcal{C} \subset A$ thus points in \mathcal{C} eventually map to D_{attr} under iteration of f , and $f(\mathcal{C}) \subset \mathcal{C}$. Therefore \mathcal{C} has points in common with \mathcal{C}' . Since \mathcal{C} is a connected component of the preimage by Φ_{attr} of the line $\Phi_{\text{attr}}(v) + [0, +\infty[$ and \mathcal{C}' is a connected subset of the latter, \mathcal{C} has to contain \mathcal{C}' .)

We will also require $r_0 < |v|$. Then the set $\mathcal{C}' \subset D_{\text{attr}}$ does not cross the circle of equation $|z| = |v|$. Now \mathcal{C} is the union of \mathcal{C}' and of a connected component of the preimage by $\Phi_{\text{attr}} = \Phi_{\text{attr}}[f]$ of the segment $\Phi_{\text{attr}}(v) + [0, n_0]$. As f varies in \mathcal{F} , the maps $\Phi_{\text{attr}} - \Phi_{\text{attr}}(v)$ all have an inverse branch defined on a common open connected neighborhood V of the segment $S = [0, n_0]$, mapping $0 = \Phi_{\text{attr}}(v) - \Phi_{\text{attr}}(v)$ back to v . This family is normal (there are many reasons for this; for instance one can use continuity of $f \mapsto \Phi_{\text{attr}}[f]$ together with compactness of \mathcal{F} ; or remark that it is 1-Lipschitz, hence equicontinuous, from the hyperbolic norm on the chosen neighborhood V of the segment S to the metric $|dz|/4|z|$, because it maps in the simply connected set A that avoids 0 so one can use the Schwarz-Pick inequality). It also avoids 0. Take a lift $\tilde{\mathcal{C}}$ of \mathcal{C} by $E : z \mapsto e^{2\pi iz}$. This is a curve starting from a preimage \tilde{v} of v and ending at ∞ tangentially to a vertical line. The part corresponding to \mathcal{C}' lives in the upper half plane “ $\text{Im}(z) > \text{Im}(\tilde{v})$ ” because we took $r_0 < |v|$. The rest is the image of S by a normal family defined in V . In particular it has bounded Euclidean length. Let L_1 be a bound.

There are infinitely many connected component of $f^{-1}(\mathcal{C})$. Consider any of them. It consists either in a single curve or in a union of d curves starting from a common critical point of f . Each of these curves has a part mapped in $\mathcal{C} \setminus \mathcal{C}'$ by f that has box-Euclidean length $\leq L_1$, and a part mapped to \mathcal{C}' by f that is completely contained in one box. By Lemma 30, the union of cuboxes visited by the full curve has

$$(4) \quad \text{combinatorial diameter} \leq \lfloor L_1 \rfloor + 1.$$

The lifted immediate basin \tilde{A} contains exactly one component of $E^{-1}f^{-1}(\mathcal{C})$ and is disjoint from all other components. We claim that A is contained in $\mathcal{B}_{\lfloor L_1 \rfloor + 2}$: indeed consider the union G_1 of the $2d - 1$ cuboxes which contain the critical point in A . It is contained in \mathcal{B}_1 . The immediate basin A contains exactly one component of $f^{-1}(\mathcal{C})$. Let G_2 be the component containing A of the complement in U_1^* of the union of all other components of $f^{-1}(\mathcal{C})$ (Figure 27 may help). It is enough to prove that G_2 is contained in $\mathcal{B}_{\lfloor L_1 \rfloor + 2}$.

The boundary of G_2 in U_1^* consists in curves all of whose starting points s are preimages of v . We claim that they all belong to \mathcal{B}_1 . Indeed the curve \mathcal{C} is isotopic in \mathbb{C} to the straight segment from v to 0 by an isotopy that does not move its endpoints. This isotopy extends to the whole Riemann sphere into an isotopy fixing ∞ . The singular values of f are $\{0, v, \infty\}$ and thus the isotopy does not

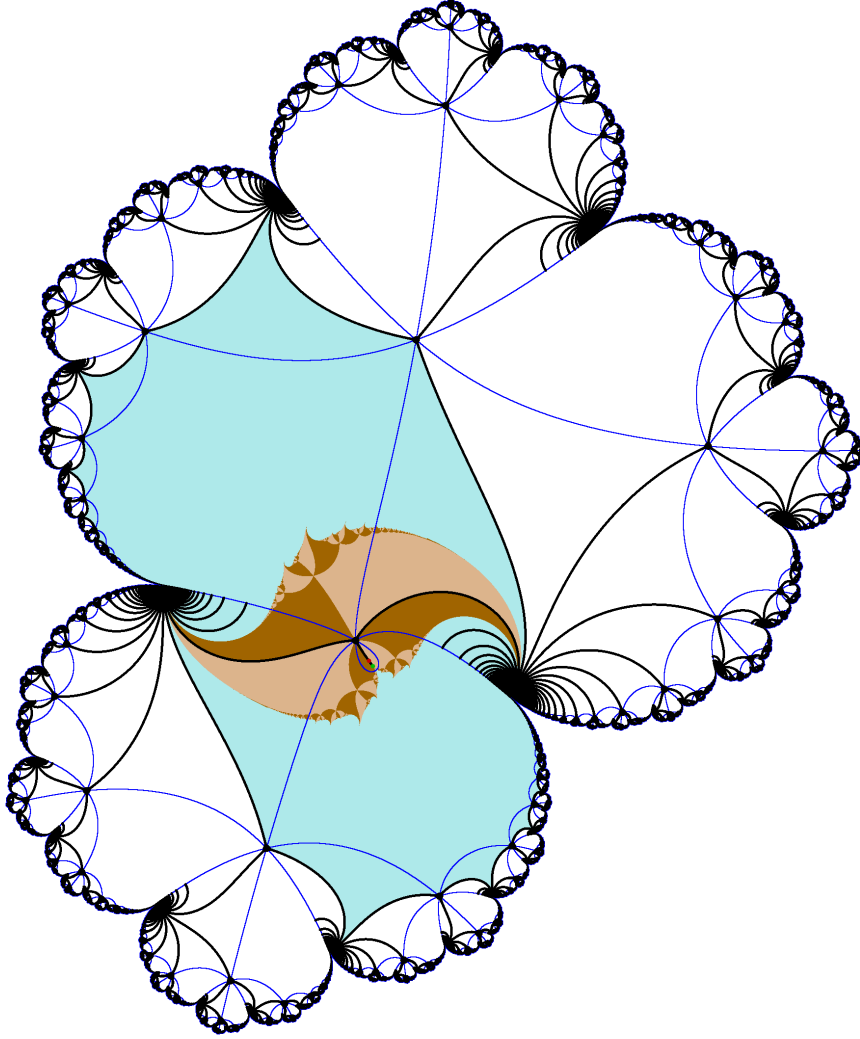


Figure 27: Example for $d = 3$. We chose some $f \in \mathcal{F}$ (more precisely we took the first renormalization of $z \mapsto z^3 + c$ with c so that there is a fixed parabolic point). The blue graph is the structural chessboard of f . The origin is marked by a tiny green dot and the critical value of f by a red one. It is a parabolic point of f . We drew in brown shades the dynamical chessboard of f in the immediate basin A of this point. The dark lines are the set $f^{-1}(\mathcal{C})$ where \mathcal{C} is the principal curve (see the text). The light blue set is the component containing A of $\text{Dom}(f)$ minus the all the dark lines that are not contained in A . The picture has been accurately drawn, the curve \mathcal{C} is a small curve part in black graph, from the green dot to the red one. It is very close to be a segment. As a consequence, $f^{-1}(\mathcal{C})$ is formed of curves that are very close to intrinsic verticals of cuboxes. It seems therefore that the light blue is completely contained in \mathcal{B}_2 . This is probably the case for all maps in \mathcal{F} for $d = 3$ because the loop is very small. It may still hold when d gets close to ∞ , but that would require a more detailed specific analysis as in [IS04], starting from the fact that v_f is close to 0 ($|v_f| \in [\sim 1/140, \sim 1/20]$, see page 25 in Section 2.3). We decided instead to resort to general arguments instead: in the proof of Lemma 31 we consider cases where \mathcal{C} may be very far from a segment.

move the singular values of f . Hence the extended isotopy lifts by f to an isotopy of U_1 . This lifted isotopy does not move the points in $f^{-1}(v)$. Now a starting point s as above can be linked to the unique critical point $c_0 \in b_*$ by a path within G_2 . The lifted isotopy deforms this path into a path with the same endpoints and that is completely contained in the complement of $f^{-1}([0, v])$. The image by f of the new path is contained in $\mathbb{C} \setminus [0, v]$ and goes from v to v . It is homotopic to a path completely contained $|z| > 1$. The homotopy lifts by f . Hence s and c_0 are linked by a path contained in a cubox. Whence the claim.

Consider any point $z \in G_2$. If z belongs to $f^{-1}(\mathcal{C})$ then it belongs to the unique component of $f^{-1}(\mathcal{C})$ in G_2 , which is the one attached to the critical point in A , which belongs to b_* . Hence $z \in \mathcal{B}_{\lfloor L_1 \rfloor + 1}$ by the bound (4) above. Otherwise, $f(z) \notin \mathcal{C}$. Then $f(z) \in H$ for $H = \mathbb{D} \setminus \{0\}$ or $H = \mathbb{C} \setminus \mathbb{D}$ (if $|f(z)| = 1$ then either can be chosen). There is a path $\gamma \subset H$ from $f(z)$ to a point of $\mathcal{C} \setminus \{0\}$ (which may be its endpoint v). Let b be the (unique) cubox containing z and such that $f(b) = H$. The path γ lifts by f to a path within b from z to a point w on the boundary of G_2 . The point w belongs thus to a component of $f^{-1}(\mathcal{C})$. We saw that this component is attached to a point in $f^{-1}(v)$ that belongs to \mathcal{B}_1 . By the bound (4), we get that $b \in \mathcal{B}_{\lfloor L_1 \rfloor + 2}$. This ends the proof that $G_2 \subset \mathcal{B}_{\lfloor L_1 \rfloor + 2}$. \square

Let the combinatorial distance between two points of $\text{Dom } \tilde{f}$ be the smallest combinatorial distance of boxes containing them. Two important facts used in the following lemma are that the chessboard graph of $\text{Dom } \tilde{f}$ is a tree and that the boundary in $\text{Dom } \tilde{f}$ of a \tilde{f} -cubox is a connected subset of this graph.

Lemma 32. *For any two points $w, z \in \text{Dom } \tilde{f}$ then there is a unique shortest path γ' from w to z for the box-Euclidean distance. If m denotes the combinatorial distance from w to z then γ' can be cut into $\leq m + 1$ connected pieces, each of which stays in some cubox.*

Proof. ²⁶ Let us define a projection from $\text{Dom } \tilde{f}$ to the chessboard graph of \tilde{f} as follows. Recall that each \tilde{f} -cubox b is homeomorphically mapped by \tilde{f} to a closed half plane and that the box-Euclidean metric element is sent to the canonical Euclidean element $|dz|$ of \mathbb{C} . The vertical projection on this half plane is 1-Lipschitz and can be conjugated back to a projection from b to its boundary in $\text{Dom } \tilde{f}$. The union of all these projections for all cuboxes b is easily seen to match at the boundary points and corners, and yields a projection function from $\text{Dom } \tilde{f}$ to the chessboard graph, which is locally 1-Lipschitz for the box-Euclidean metric (the only place where checking this claim is not trivial is at corners), and hence globally because box-Euclidean distance is defined by minimizing path length.

Given any path γ from w to z , if the path meets the chessboard graph then the part from its first intersection with the graph to its last can be projected as above. The new path is strictly shorter unless the part was already contained in the graph. This part can be further simplified into an injective path within the graph, strictly shorter unless it was already injective.

If moreover both w and z belong to a given cubox c , the first and last point in the graph are also in c , and since the graph is a tree and the boundary of a cubox is a connected subset of this tree, the simplified part is necessarily contained in this boundary, hence the simplified path is contained in c . We have thus in particular proved that for any path that is not completely contained in c there is a strictly shorter path contained in c . Hence the straight segment γ'' from w to z for the Euclidean structure on c is the unique shortest box-Euclidean path from w to z .

²⁶Special thanks to Arnaud Mortier for a great help in this proof.

within $\text{Dom } \tilde{f}$. The other conclusions of the lemma are trivial in this case: $m = 0$ and γ'' does not need to be cut.

In the rest of the proof of the lemma, we assume that there is no cubox containing both w and z .

Then, given the simplification of path constructed above, it follows that the infimum of box-Euclidean lengths of paths between w and z is the same as the infimum over the set \mathcal{A} of paths defined below, and that a path that is not in \mathcal{A} cannot be minimal. The set \mathcal{A} consist in paths that are a straight box-Euclidean line from w to the boundary of its box if w is in the interior of a box, then an injective path within the graph, then similarly a straight box-Euclidean line to z if z is in the interior of a box. From the form of \mathcal{A} and the fact that the distance along the graph between two points a and b of the graph is a continuous function of the pair (a, b) , the fact that a minimal distance is reached on \mathcal{A} easily follows. Let us sum up what we have proved so far: there is at least one shortest path, all shortest paths are in \mathcal{A} .

Let I_w be defined as follows: if w is in the graph we let $I_w = \{w\}$; otherwise we let I_w be the boundary in $\text{Dom } \tilde{f}$ of the unique cubox containing w . In the latter case, I_w is an infinite curve in the graph. The set $I_w \cap I_z$ is either empty or a point or a connected curve, of finite or infinite length.

If $I_w \cap I_z$ is empty or a point, then there is a unique shortest path γ'' within the graph from I_w to I_z ; we allow γ'' to be reduced to a single point to include the case when $I_w \cap I_z$ is a single point. We call w' the initial point of γ'' and z' the endpoint; as we explained, z' may be equal to w' . It is also possible that $w = w'$, similarly $z = z'$ is possible. If $w \neq w'$ then there is a unique cubox containing both. Similarly for z and z' . In all cases, the box-straight path from w to w' , followed by γ'' , followed by the box-straight path from z' to z is the unique shortest path in \mathcal{A} , and thus the unique shortest path within $\text{Dom } \tilde{f}$. Call it γ' .

Consider now any cubox chain b_0, \dots, b_m with $w \in b_0$ and $z \in b_m$. This chain necessarily covers γ' . Let us prove that claim. Note that the part of γ' from w to w' is contained b_0 and the part from z' to z in b_m . For the rest of γ' , let us now work by contradiction and assume that some point u on γ'' does not belong to the union of the b_k . In other words, none of the cuboxes containing u is one of the b_k . Now consider the union X of all cuboxes that do not contain u . It is a closed set containing all the b_k . Since the b_k form a connected chain from w' to z' , they are all contained in the same connected component of X . Consider path from w' to z' within this component, project it on the graph and simplify as above. This leads to an injective path from w' to z' , contained in the same component of X and contained in the graph. By uniqueness of injective paths in a tree, this path would go through u , leading to a contradiction.

Let us now split γ' as follows: choose any cubox b_i containing w , define $i_1 = i$ and cut γ' at the last point where it is contained in b_{i_1} . If this cutpoint is not the endpoint of γ' , then a non-trivial sub-part of the path starting from b_{i_1} belongs to another cubox $b_{i'}$. Define $i_2 = i'$ and cut the remaining part of the path at the last point where it is contained in b_{i_2} . And so on. This process necessarily ends (because, for instance, the cut points are contained in a discrete set, because they are either branch points of the graph or the first or the last intersection of γ' with the graph). So we get a finite sequence of cuboxes $b_{i_1}, b_{i_2}, \dots, b_{i_{m'}}$ for some $m' \in \mathbb{N}^*$ and a splitting $\gamma'_1, \dots, \gamma'_{m'}$ of γ into connected pieces with $\gamma'_j \subset b_{i_j}$ for all $j \leq m'$. By construction $b_{i_{j+1}} \neq b_{i_j}$. Now since the graph is a tree, hence there is a “unique” injective path in the graph between two points, and since the intersection of a cubox with the graph is a connected subset of this tree (it is a curve, infinite in both directions), it follows that no two cuboxes b_{i_j} and b_{i_k} can

be equal for $j \geq k + 2$, for otherwise the whole part of the path between γ'_j and γ'_k would be contained in this cubox, contradicting the way we built the splitting. Hence $m' \leq m + 1$.

The last case is when $I_w \cap I_z$ is a connected curve in the graph. Let b be the unique cubox containing w and b' be the same for z . Note that b and b' are adjacent, and $m = 1$. The union $b \cup b'$ is connected. It consists in the interior of b , the interior of b' , the common curve, and at most four disjoint pieces of curves in the boundaries of b or b' , attached to an end point of the common curve. Because the graph is a tree, all paths in \mathcal{A} are contained in $b \cup b'$ and all paths in \mathcal{A} must meet the common curve, possibly at an end thereof. It follows that the shortest path in \mathcal{A} from w to z is a straight segment to a point in the common curve, followed by a straight segment. We have thus cut the shortest path in two pieces satisfying the conclusion of the lemma, since $m + 1 = 2$. \square

Let us now go back to the situation we were studying: recall $L(\varepsilon)$ was defined at the beginning of Section 3.6; for convenience we denote $L(\varepsilon)$; we had a path γ of A -length at most $c_2 + L$, starting from Mudba and going to some point z_0 . We are ready to prove that:

$$(5) \quad d_{U_1^*}(\gamma(0), \gamma(1)) \leq c'_6 + c_6 \log(1 + L).$$

Recall that there is a special cubox b_* that is a punctured neighborhood of the origin. Note that b_* is the only cubox that has a unique lift by E , which we denote \tilde{b}_* . Let $\tilde{\gamma}$ be a lift of γ by E and $\gamma_2 = \tilde{f} \circ \tilde{\gamma}$. Then γ_2 is also a lift of $f \circ \gamma$ by E . The U_1^* -hyperbolic length of γ is equal to the $\text{Dom}(\tilde{f})$ -hyperbolic length of $\tilde{\gamma}$. The box-Euclidean length of $\tilde{\gamma}$ is equal to Euclidean length of γ_2 and is thus $\leq 2(c_2 + L)$ by Equation (2). By Lemma 31, the path γ is contained in \mathcal{B}_{c_4} . There is thus a chain of cuboxes of length at most c_4 from some cubox containing $\gamma(0)$ to b_* . This chain lifts by E into a chain of cuboxes from $\tilde{\gamma}(0)$ to \tilde{b}_* . Similarly, there is a chain of length $\leq c_4$ from b_* to $\gamma(1)$ and it lifts to a chain from \tilde{b}_* to $\tilde{\gamma}(1)$. Hence the combinatorial distance²⁷ from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$ is $\leq 2c_4$. Consider the path γ' provided by Lemma 32, from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$, of box-Euclidean length at most that of $\tilde{\gamma}$, and consisting in $p \leq 2c_4 + 1$ parts γ'_i each contained in some cubox. Denote L_i the box-Euclidean length of γ'_i . Then $\sum_{i=1}^p L_i \leq 2(c_2 + L)$ and in particular $L_i \leq 2(c_2 + L)$. By Lemma 28, the endpoints of γ'_i sit at U_1^* -hyperbolic distance $\leq c'_5 + \log(1 + c_5 L_i)$. Thus, putting it all together:

$$\begin{aligned} d_{U_1^*}(\gamma(0), \gamma(1)) &\leq d_{\text{Dom} \tilde{f}}(\tilde{\gamma}(0), \tilde{\gamma}(1)) \\ &\leq \sum_{i=1}^p c'_5 + \log(1 + c_5 L_i) \\ &\leq p c'_5 + p \log(1 + 2c_5(L + c_2)) \\ &\leq (2c_4 + 1)c'_5 + (2c_4 + 1) \log(1 + 2c_5 L + 2c_5 c_2) \\ &\leq c'_6 + c_6 \log(1 + L) \end{aligned}$$

for some constants c_6, c'_6 that depend only on c_2, c_4, c_5 and c'_5 , which proves (5).

Because of the inclusion $U_1^* \subset U_1$, the U_1 -hyperbolic distance between $\gamma(0)$ and $\gamma(1)$ will be even shorter. Using Lemma 25, we get that z_0 belongs to the U_1 -hyperbolic ball of center 0 and radius

$$L' = c_7 + c'_6 + c_6 \log(1 + L).$$

Hence the set C object of Proposition 22, which we are proving, is contained in this hyperbolic ball (see the discussion after Lemma 25). Recall that $L = L(\varepsilon) =$

²⁷notion defined just before Lemma 32

$\tanh^{-1}(1 - \varepsilon)$. Introduce $\varepsilon' \in]0, 1[$ such that $\tanh^{-1}(1 - \varepsilon') = L'$. Then

$$C \subset \phi_1(B(0, 1 - \varepsilon')).$$

Now $\varepsilon' = 2/(e^{2L'} + 1) \geq e^{-2L'}$ and $L' = c_7 + c'_6 + c_6 \log(1 + L)$ and $L \leq c_1 + \frac{1}{2} \log \frac{1}{\varepsilon}$ so $L' \leq c'_8 + c_8 \log(1 + \log(1/\varepsilon))$, thus

$$(6) \quad \boxed{\log \frac{1}{\varepsilon'} \leq c'_9 + c_9 \log \left(1 + \log \frac{1}{\varepsilon} \right)}$$

In particular, as $\varepsilon \rightarrow 0$, ε' also tends to 0 but remains much bigger than ε . This proves Proposition 22.

3.7. Step 2, I: Perturbation argument. Let us recall the notations introduced in Section 3.2:

$$\mathcal{F} = \{ \mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : \mathbb{D} \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2) \}$$

and

$$\mathcal{F}_\varepsilon = \{ \mathcal{R}[B_d] \circ \phi^{-1} \mid \phi : B(0, 1 - \varepsilon) \rightarrow \mathbb{C} \text{ is univalent and } \phi(z) = z + \mathcal{O}(z^2) \}$$

where $\mathcal{R}[B_d]$ is the (upper) parabolic renormalization of the Blaschke product, normalized to be defined on the unit disk. In particular,

$$\mathcal{F}_0 = \mathcal{F}.$$

Last, for $X \subset [0, 1]$, we will denote

$$\mathcal{F}_X = \bigcup_{x \in X} \mathcal{F}_x.$$

3.7.1. An interpolation. Let $\varepsilon_1 > 0$ and $f \in \mathcal{F}_{\varepsilon_1}$:

$$f = \mathcal{R}[B_d] \circ \tilde{\phi}^{-1}$$

For convenience, we will denote

$$r' = 1 - \varepsilon_1$$

and $\phi(z) = \frac{1}{r'} \tilde{\phi}(r'z)$. Then $\phi \in \mathcal{S}$ (the class of Schlicht maps) and

$$f(z) = \mathcal{R}[B_d](r'\phi(z/r')).$$

Let $U = \phi(\mathbb{D})$. We will interpolate smoothly between f , which belongs to $\mathcal{F}_{\varepsilon_1}$, and an element of \mathcal{F} as follows: for $t \in [0, 1[$, let

$$\phi_t(z) = r_t \phi(z/r_t) \text{ with } r_t = 1 - t.$$

Then the map ϕ_t is an isomorphism from $B(0, r_t)$ to $r_t U$. Let

$$f_t(z) = \mathcal{R}[B_d] \circ \phi_t^{-1} : r_t U \rightarrow \mathbb{C}.$$

Then

$$f_{\varepsilon_1} = f,$$

and

$$f_t \in \mathcal{F}_t \text{ thus } f_0 \in \mathcal{F}.$$

In the sequel, we will start from knowledge about f_0 and transfer it to f_{ε_1} , by continuously increasing t from 0 to ε_1 .

Using the language of structures that we introduced in Section 1.1, let us stress that maps in \mathcal{F}_t are all $(I, \widehat{\mathbb{C}})$ -structurally equivalent (I being a singleton and the origin being the marked point). For $t' > t$, the structure of maps in $\mathcal{F}_{t'}$ is a sub-structure of that of maps in \mathcal{F}_t .

Remark. Though, for $t' > t$, $f_{t'}$ is a sub-structure of f_t , it is very unlikely that the map $f_{t'}$ would be conjugate to a restriction of f_t .

Let us show a non-commuting diagram that the reader may find useful in order to follow the arguments.

$$\begin{array}{ccc} & \xleftarrow{\cdot/r_t} & \\ \phi_0^{-1} \downarrow & & \uparrow \mathcal{R}[B_d] \\ & \xrightarrow{r_t \times \cdot} & \end{array}$$

The map f_t consists in turning once around this diagram, starting from the upper right corner.²⁸

3.7.2. About the critical value. Let T_0 be one minus the absolute value of the critical point of $\mathcal{R}[B_d]$ that is closest to 0. Then for all $t \in [0, T_0[$, maps in \mathcal{F}_t have a unique critical value.

Lemma 33. *There exists $T'_1 \in]0, T_0[$ such for all maps $f \in \mathcal{F}_{[0, T'_1]}$, the critical value is attracted to 0.*

Proof. By Fatou's theorem (Theorem 3), this is the case for all maps in \mathcal{F}_0 . The existence of T'_1 then follows from compactness of \mathcal{F}_0 and the fact that for a parabolic map with one petal attracting a given point, nearby parabolic maps will attract nearby points. \square

A consequence of the uniqueness of the critical value is that the extended attracting Fatou coordinate $\Phi_{\text{attr}}[f_t]$ has a set of critical values contained in $\{v' - n \mid n > 0\}$ where $v' = \Phi_{\text{attr}}[f_t](v)$ and v is the critical value of f_t . Unlike the case $t = 0$, when $t > 0$ the map $\Phi_{\text{attr}}[f_t]$ probably has a big set of asymptotic values (it is likely that it contains curves).

3.8. Step 2, II: Following fibers.

3.8.1. A motion of the fibers of the Fatou coordinates and of the renormalized map. The point of view outlined in Section 3.7.1 can be reversed and we may start from any map $f_0 = \mathcal{R}[B_d] \circ \phi_0^{-1} \in \mathcal{F}$, which has the full structure of $\mathcal{R}[B_d]$ and perturb it into the map $f_t \in \mathcal{F}_t$ as before, which has less and less structure as $t \in [0, 1[$ increases. Let us recall how f_t is defined:

$$f_t(z) = \mathcal{R}[B_d] \circ \phi_t^{-1} \quad \text{with} \quad \phi_t(z) = r_t \phi_0(z/r_t) \quad \text{and} \quad r_t = 1 - t.$$

Studying the survival of (part of) the structure of the parabolic renormalization $\mathcal{R}[f_t]$ as t increases means following fibers of $\mathcal{R}[f_t]$.

Recall that $\mathcal{R}[f_t]$ is defined by

$$(a^{-1} \circ \mathcal{R}[f_t] \circ b) \circ E = E \circ (\Phi_{\text{attr}}[f_t] \circ \Psi_{\text{rep}}[f_t])|_{W_t}$$

with $E(z) = e^{2\pi iz}$, W_t is some domain, and a and b are linear maps that depend on f_t and on normalization conventions. Recall that we chose to normalize Fatou coordinates by their expansion at infinity, and to normalize $\mathcal{R}[f_t]$ by fixing its critical value. See Section 3.3 for more details.

To lighten the expressions, let us abbreviate $R_t = \mathcal{R}[f_t]$ and introduce extended Fatou coordinates Φ_t and Ψ_t of f_t , normalized differently from $\Phi_{\text{attr}}[f_t]$ and $\Psi_{\text{rep}}[f_t]$, and so that

$$R_t \circ E = E \circ \Phi_t \circ \Psi_t|_{W_t}.$$

We defined in Section 3.7.2 two constants T_0 and $T'_1 < T_0$ such that:

²⁸It may at first seem to be better to start from the upper left corner, since the corresponding composition has a domain U that does not depend on t . However, when we iterate these maps, we basically go in round circles along a non-commuting diagram again and again, and the author thinks that it would not simplify the proof that much.

- For $t \leq T_0$, for all $f \in \mathcal{F}$, f_t has a unique critical value. Let us denote it by v_t .
- For $t \leq T'_1$, this point v_t is in the domain of definition of $\Phi_{\text{attr}}[f_t]$.

Let

$$\Phi_t(z) = \Phi_{\text{attr}}[f_t](z) + \beta_t$$

where $\beta_t = \Phi_{\text{attr}}[f_0](v_0) - \Phi_{\text{attr}}[f_t](v_t)$, so that $\Phi_t(v_t)$ does not depend on t . For the repelling inverse Fatou coordinate (whose normalization is less important) we let

$$\Psi_t(z) = \Psi_{\text{rep}}[f_t](z - \beta'_t),$$

for $\beta'_t = \beta_t - i\pi\gamma[f_t]$ (recall γ is the iterative residue, see Section 1.2). Let $\Phi : (t, z) \mapsto \Phi_t(z)$, that we define on

$$\text{Dom } \Phi = \{(t, z) \in [0, T'_1[\times \mathbb{C} \mid z \in \text{Dom}(\Phi_t)\}.$$

It is an open subset of $[0, T'_1[\times \mathbb{C}$, and Φ is a continuous function of (t, z) by Proposition 11 (in fact, it is analytic, see [Tan00]). Similarly, let

$$R : \begin{cases} \text{Dom } R & \rightarrow \mathbb{C} \\ (t, z) & \mapsto R_t(z) \end{cases}$$

The domain of R is an open subset of $[0, T'_1[\times \mathbb{C}$ and R is continuous, analytic w.r.t. z for fixed values of t . (It is also analytic w.r.t. (t, z) but we will not use this fact.)

The critical values of Φ_t and R_t do not move when t varies (even when some critical points vanish). It has the following consequence:

Proposition 34 (following part of the structure). *Let $F = \Phi$ or $F = R$. Then*

- (Lemma 36) *fibers of Φ form a foliation that is locally parallelizable over the first coordinate.*

It follows that there exists a function $\tau : \text{Dom } F_0 \rightarrow]0, T'_1]$ and function $\zeta(t, z)$ such that:

- $\text{Dom } \zeta = \{(t, z) \in [0, T'_1[\times \text{Dom}(F_0) \mid t \in [0, \tau(z)[\}$
- *the map τ is lower semi continuous, i.e. for all $t \in [0, T'_1[$, the set $U_t = \tau^{-1}(]t, T'_1]) \subset \mathbb{C}$ is open*
- *the above two points imply that $\text{Dom } \zeta$ is an open subset of $[0, T'_1[\times \mathbb{C}$ and $\text{Dom } \zeta = \{(t, z) \in [0, T'_1[\times \mathbb{C} \mid z \in U_t\}$*
- *the map ζ is continuous*
- *for each fixed $t \in [0, T'_1[$, the map $z \in U_t \mapsto \zeta(t, z)$ is holomorphic and injective*
- $\forall (t, z) \in \text{Dom } \zeta$, $F_0(z) = F_t(\zeta(t, z))$, i.e. *the map $t \in [0, \tau(z)[\mapsto (t, \zeta(t, z))$ follows a fiber of F*
- (uniqueness) *any continuous map following a fiber of F as t varies from 0 to some t_0 , starting from $(0, z) \in \text{Dom } F$, must coincide with $t \in [0, t_0] \mapsto \zeta(t, z)$.*

The rest of the present section is devoted to the proof of the above proposition. The proof is written for Φ but is the same, word for word, for R .

Lemma 35. *Let $(t_0, z_0) \in \text{Dom } \Phi$ and assume that z_0 is a critical point of Φ_{t_0} . Then there exists a connected neighborhood I of t_0 in $[0, T'_1[$, and $r_0 > 0$ such that for all $t \in I$, Φ_t has a unique critical point in $B(z_0, r_0)$, it moves continuously with t and its multiplicity does not change.*

Proof. We apply Hurwitz's theorem²⁹ to Φ'_t and to Φ_t (note that Φ'_t also depends continuously on t , by Cauchy's estimates): let $u = \Phi_{t_0}(z_0)$. Take $r_0 > 0$ small enough so that z_0 is the only critical point of Φ_{t_0} in $\overline{B} := \overline{B}(z_0, r_0)$, the only solution of $\Phi_{t_0}(z) = u$ in \overline{B} , and such that Φ_{t_0} maps this disk in $B(u, 1/2)$; there exists ε_0 such that for all $t \in [0, T'_1[$ with $|t - t_0| < \varepsilon_0$, Φ_t is defined on \overline{B} and maps it in $B(u, 1/2)$; then by Hurwitz's theorem, there exists $0 < \varepsilon < \varepsilon_0$ such that for $|t - t_0| < \varepsilon$, $\Phi_t - u$ has $d - 1$ critical points counted with multiplicity in B and d roots in B . Now recall we normalized the maps Φ_t so that all critical values belong to $\mathbb{Z} + u$ and u does not depend on t . Since $\Phi_t(\overline{B}) \subset B(u, 1/2)$, this implies that all critical points of Φ_t in \overline{B} map to u . Thus the sum of local degrees of Φ_t at preimages of u in \overline{B} equals d , and the sum of local degrees minus one equals $d - 1$: there is exactly one preimage of u , thus exactly one critical point. Moreover, its local degree is d , thus its multiplicity is constant. Continuous dependence is a classical application of Hurwitz's theorem and is left to the reader.³⁰ \square

Now consider the fibers of Φ : $X_c = \{(t, z) \in \text{Dom } \Phi \mid \Phi(t, z) = c\}$. They form a collection of disjoint closed subsets of $\text{Dom } \Phi$. We will prove that this collection is a locally trivial foliation, in the following precise sense:

Lemma 36 (local trivialization). *All $(t_0, z_0) \in \text{Dom } \Phi$ has an open neighborhood V in $\text{Dom } \Phi$ on which a change of variable $U : V \rightarrow V' \subset [0, T'_1[\times \mathbb{C}$ of the form*

$$U : (t, z) \mapsto (t, u(t, z))$$

is defined,

- (1) U is a homeomorphism to V' ,
- (2) for all $t, z \mapsto u(t, z)$ is holomorphic,
- (3) $\forall c \in \mathbb{C}$, $U(X_c)$ is the intersection of a horizontal with V' : it is of the form $V' \cap ([0, T'_1[\times \{w\})$ for some $w \in \mathbb{C}$.

Proof. Case 1: z_0 is not a critical point of Φ_{t_0} . It is an application of Hurwitz's theorem. Since the family Φ_t depends continuously on t and Φ_{t_0} is not locally constant near z_0 , one can deduce from Hurwitz's theorem that the map $U = \Phi$ itself, restricted to an appropriate neighborhood V , will be a local trivialization. Details are left to the reader.

Case 2: z_0 is a critical point of Φ_{t_0} . A consequence of Lemma 35, is that $\Phi_t(z) = (z - c_t)^d h_t(z)$ where $h_t(z)$ is a holomorphic function in z , continuous in (t, z) , defined locally and non-vanishing. The map $g(t, z) = \sqrt[d]{h_t(z)}$ is defined locally, and we leave to the reader to check that the map $(t, z) \mapsto (t, (z - c_t)g(t, z))$ is a local trivialization. \square

Hence connected components of fibers are graphs of continuous functions $t \mapsto z(t)$ defined on connected open subsets of $[0, T'_1[$. Now, given any $z \in \text{Dom}(\Phi_0)$, we follow its fiber as t increase from 0 as long as possible: this gives a maximal continuous function $t \in [0, \tau(z)[\mapsto \zeta_z(t)$ such that $\zeta_z(0) = z$ and $\Phi_t(\zeta_z(t))$ is constant. The real

²⁹There seems to be several statements called Hurwitz's theorem. We are referring to the following: for a sequence of holomorphic functions f_n converging uniformly on compact subsets of an open subset U of \mathbb{C} , call its limit f . If D is a disk compactly contained in U and f does not vanish on the boundary of D then for all n big enough, f and f_n have the same number of zeroes in D , counted with multiplicity.

³⁰There is a more direct proof, with Hurwitz's theorem used only at the end to deduce continuity. From the fact that z_0 is in a parabolic basin and that all critical points of f_t map to the same point, it follows that the orbit of z_0 hits the set of critical points only once. Then one uses that $\Phi_{\text{attr}} = -n + \Phi_{\text{attr}} \circ f^n$, and that Φ_{attr} is injective in the petal $D_{\text{attr}}[f_t]$ and that the latter moves continuously with t . Similar arguments can be carried out for R in place of Φ .

number $\tau(z)$ belongs to $]0, T'_1]$. Uniqueness (last point of Proposition 34) follows easily. In the lemma below, the point $\zeta_z(t)$ is denoted

$$z\langle t \rangle.$$

Lemma 37. *The following holds:*

- (1) *The function τ is lower semi-continuous, i.e. for all $t \in [0, T'_1]$, the set $U_t = \tau^{-1}(]t, T'_1]) \subset \mathbb{C}$ is open.*
- (2) *On U_t , the function $z \mapsto z\langle t \rangle$ is holomorphic.*

Proof. For a given $z \in U_t$, since $t < \tau(z)$, cover the compact set $[0, t]$ by open subsets on which there is a local trivialization of the fiber z belongs to. Extract a finite cover. From it, one can build a trivialization like in the previous lemma, but in a whole neighborhood of $z\langle [0, t'] \rangle$ relative to $[0, t'] \times \mathbb{C}$. The lemma follows. \square

This ends the proof of the Proposition 34.

3.8.2. Objectives. Let $f \in \mathcal{F}$ and denote by $\tau_R[f]$ the τ function corresponding to R in Proposition 34: i.e. $\tau_R[f](z)$ is the time up to which the fiber of $(t, z) \mapsto R_t(z)$ that contains $(0, z)$ can be followed. Recall that R_t denotes the parabolic renormalization of f_t , normalized so that the critical value does not move as t varies, and recall that f_t is a specific perturbation of $f_0 = f$. Consider the parabolic renormalization $\mathcal{R}[f_0]$ of f_0 .

Lemma 38. *If $\forall z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$, $\tau_R[f_0](z) > \varepsilon_0$ then $\mathcal{R}[f_{\varepsilon_0}]$ has a restriction that belongs to $\mathcal{F}_{\varepsilon_1}$.*

Proof. The map $\mathcal{R}[f_0]$ belongs to \mathcal{F} , thus it can be written as $\mathcal{R}[f_0] = \mathcal{R}[B_d] \circ \phi_2^{-1}$ where $\phi_2 : \mathbb{D} \rightarrow \mathbb{C}$ is univalent and $\phi_2(z) = z + \mathcal{O}(z^2)$. By hypothesis, the set U_{ε_0} contains $\text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1) = \phi_2(B(0, 1 - \varepsilon_1))$ (the sets U_t were defined in Lemma 37). According to Proposition 34, the map $\zeta^t : z \in U_t \mapsto \zeta(t, z)$ is a holomorphic bijection to its image, and $\mathcal{R}[f_t](\zeta^t(z)) = \mathcal{R}[f_0](z)$ holds on U_t . Apply this to $t = \varepsilon_0$: let $V = \zeta^{\varepsilon_0}(\phi_2(B(0, 1 - \varepsilon_1)))$, then $\zeta^{\varepsilon_0} \circ \phi_2$ is a structural equivalence, with 0 as a marked point, between the restriction of $\mathcal{R}[f_{\varepsilon_0}]$ to V and the restriction of $\mathcal{R}[B_d]$ to $B(0, 1 - \varepsilon_1)$. \square

So the Main theorem will be proved if we can prove the following claim:

Assertion 39 (survival of fibers of R). *There exists a pair $\varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 < T'_1$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$,*

$$\tau_R[f_0](z) > \varepsilon_0.$$

We will in fact prove more: for all ε_0 small enough, there exists $\varepsilon_1 < \varepsilon_0$ such that the conclusion of the assertion holds. Better: we can take $\varepsilon_1 \ll \varepsilon_0$ (see details in Section 3.10).

3.8.3. Restatement of the objectives. Let $\varepsilon > 0$ and consider some

$$z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon).$$

The map R_t is the semi-conjugate by E of the composition $\Phi_t \circ \Psi_t$, but it can also be viewed differently: recall that the extended Fatou coordinates Φ_t and extended inverse Ψ_t are defined via iteration of f_t , using bijective Fatou coordinates in petals as a starting point. Let \mathcal{P}_{rep} be a repelling petal and Φ_{rep} be a repelling Fatou coordinate such that $\Psi_t = \Phi_{\text{rep}}^{-1}$ holds on $\Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$. The value $R_t(z)$ thus decomposes as follows (see Figure 2 in Section 1.2):

$$R_t(z) = E(\Phi_t(f_t^{m_0}(\Psi_t(u))))$$

where $E(z) = e^{2\pi iz}$, $u \in E^{-1}(z) \cap \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$ and $m_0 = m_0(z) \in \mathbb{N}$ is chosen so that $f_t^{m_0}(\Psi_t(u))$ belongs to the attracting petal. Let us now focus on the initial situation, at $t = 0$: consider the f_0 bilateral orbit

$$(n \in \mathbb{Z}) \quad \omega_n := \Psi_0(u + n).$$

It depends on z and on the choice of $u \in E^{-1}(z) \cap \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$. Interestingly, if one chooses another $u \in E^{-1}(z) \cap \Phi_{\text{rep}}(\mathcal{P}_{\text{rep}})$, we get the same orbit, but with the index n shifted. According to the first step, if $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon)$ then the orbit ω_n is contained in $\text{Dom}(f_0) \odot (1 - \varepsilon') = \phi_0(B(0, 1 - \varepsilon'))$ with $\varepsilon' \gg \varepsilon$:

$$\forall n \in \mathbb{Z}, \omega_n \in \text{Dom}(f_0) \odot (1 - \varepsilon'). \quad {}^{31}$$

Let us now denote $\tau_\Phi[f]$ the τ function corresponding to Φ in Proposition 34. This proposition also provides a map $(t, z) \mapsto \zeta(t, z)$, to be interpreted as a motion of z as t varies. For convenience, in the sequel we will use the notation

$$z\langle t \rangle = \zeta(t, z).$$

Lemma 40 (the motion is compatible with the dynamics). $\forall z \in U_0$, $\tau_\Phi(f_0(z)) \geq \tau_\Phi(z)$ and $\forall t < \tau_\Phi(z)$, $f_t(z\langle t \rangle) = f_0(z)\langle t \rangle$.

Proof. By construction of the extended Fatou coordinates, if $(t, z) \in \text{Dom } \Phi$ then $(t, f_t(z)) \in \text{Dom } \Phi$ and $\Phi(t, f_t(z)) = 1 + \Phi(t, z)$. By hypothesis, the graph of $t \in [0, \tau_\Phi(z)) \mapsto z\langle t \rangle$ is contained in $\text{Dom } \Phi$ hence so is the graph of $t \in [0, \tau_\Phi(z)) \mapsto f_t(z\langle t \rangle)$ and $\Phi(t, f_t(z\langle t \rangle)) = 1 + \Phi(t, z\langle t \rangle)$, and thus remains constant as t varies, by construction of the motion $z\langle t \rangle$. This means that $t \in [0, \tau_\Phi(z)) \mapsto f_t(z\langle t \rangle)$ is in the unique fiber of Φ containing $f_0(z)$: hence $f_t(z\langle t \rangle) = f_0(z)\langle t \rangle$. \square

Now for a given t consider the sequence

$$\omega_n\langle t \rangle.$$

It is an orbit of f_t , though, depending on t , it may not be defined for all n :

Lemma 41. *For all $t \in [0, T'_1[$:*

- *if $\omega_n\langle t \rangle$ is defined (i.e. $\tau_\Phi(\omega_n) > t$) then $\omega_{n+1}\langle t \rangle$ is defined and $\omega_{n+1}\langle t \rangle = f_t(\omega_n\langle t \rangle)$,*
- *$\omega_n\langle t \rangle$ is defined when n is big enough.*

Proof. Since $\omega_n\langle 0 \rangle = \omega_n$ is an orbit for f_0 : $\omega_{n+1}\langle 0 \rangle = f_0(\omega_n\langle 0 \rangle)$. The first point follows from the previous lemma. Informally, the second point states that points deep enough in the attracting petal can be followed for a long time. Let us apply Proposition 9 and its companion Proposition 7 to the family of maps $\mathcal{G} = \{f_s \mid s \in [0, t]\}$. The Fatou coordinates in this proposition are normalized by the expansion. They thus differ from Φ_s by the constant β_s , which is bounded for $s \in [0, t]$. Hence there is a map ξ , independent of $s \in [0, t]$, such that the domain of equation $\text{Re}(z) > \xi(\text{Im}(z))$ is contained in the image by Φ_s of the attracting petal $D_{\text{attr}}[f_s]$ (defined in Proposition 7). Choose N_1 so that $\omega_{N_1}\langle 0 \rangle \in D_{\text{attr}}[f_0]$. For $N = N_1 + k \geq N_1$, we have $\omega_N\langle 0 \rangle \in D_{\text{attr}}[f_0]$ and $\Phi_0(\omega_N\langle 0 \rangle) = \Phi_0(\omega_{N_1}) + k$ hence there is some $N_2 \geq N_1$ such that for all $n \geq N_2$, $\omega_n\langle 0 \rangle$ is in the domain of equation

³¹Let us again insist on our interpretation of this fact, that is the central idea of the whole machinery: given $f_0 \in \mathcal{F}$, the restriction of its renormalized map R_0 to a maps with substructure \mathcal{F}_ε , can be defined using a restriction of the map f_0 that has structure $\mathcal{F}_{\varepsilon'}$, i.e. much less structure. If all maps with structure $\mathcal{F}_{\varepsilon'}$ were restrictions of maps in \mathcal{F} we would be done (the main theorem would follow at once), but this is of course not the case, and this is the reason why we introduced the interpolation f_t . The idea is then the following: since $\varepsilon \ll \varepsilon'$, for t at most ε or just slightly bigger, the map f_t will be extremely close to f_0 on a set slightly bigger than $\text{Dom}(f_0) \odot (1 - \varepsilon')$. The task is then to check that this is close enough so that the fibers attached to the orbits ω_n survive and thus the \mathcal{F}_ε -structure of the parabolic renormalization survives.

$\operatorname{Re}(z) > \xi(\operatorname{Im}(z))$. Let us call $\Psi_{\text{attr},s}$ the inverse of the restriction of Φ_s to the petal. The function $s \mapsto \Psi_{\text{attr},s}(\Phi_0(\omega_n\langle 0 \rangle))$ then defines a motion of $\omega_n\langle 0 \rangle$ within a fiber of Φ , whence the conclusion by the uniqueness point of Proposition 34. \square

The sequence $\omega_n\langle t \rangle$ is thus defined either for all $n \in \mathbb{Z}$ or for all $n \geq N \in \mathbb{Z}$, where N depends both on t and on the orbit $\omega_n = \omega_n\langle 0 \rangle$. Proposition 8 provides a repelling petal $D_{\text{rep}}[f_t]$ of diameter r_0 that varies continuously with f_t . Here r_0 can be any small enough constant independent of f_t . Assertion 39, and thus the main theorem, will follow from:

Assertion 42 (survival of orbits as fibers of Φ , and control). *There exists $r'_0 < r_0$ and a pair $\varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 < T'_1$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \operatorname{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$, if we consider the orbit ω_n associated to z , then*

- for all $n \in \mathbb{Z}$

$$\tau_\Phi[f_0](\omega_n) > \varepsilon_0,$$

- there exists $M \in \mathbb{Z}$ such that $(t \leq \varepsilon_0 \text{ and } n \leq M) \implies \omega_n\langle t \rangle \in D_{\text{rep}}[f_t](r'_0)$.

Indeed, recall that we defined ω_n starting from some $z \in \operatorname{Dom}(R_0) \odot 1 - \varepsilon_1$. Let $\Phi_{+,t}$ be the repelling Fatou coordinates on $D_{\text{rep}}[f_t]$ such that $\Psi_t \circ \Phi_{+,t}(z) = z$ holds on $D_{\text{rep}}[f_t]$. Let then $z(t) = E(\Phi_{+,t}(\omega_n\langle t \rangle))$. Then $z(t) \in \operatorname{Dom} R_t$ and $\forall n \geq M$, $R_t(z(t)) = E(\Phi_{+,t}(\omega_n\langle t \rangle)) = E(\Phi_t(\omega_n\langle t \rangle) + M - n) = E(\Phi_t(\omega_n\langle t \rangle)) = E(\Phi_0(\omega_n\langle 0 \rangle))$ (the last equality because we follow a fiber of Φ), i.e. $R_t(z(t))$ is constant as t varies. Since $z(0) = z\langle 0 \rangle$, we have followed the R -fiber associated to z : $z(t) = z\langle t \rangle$. In particular $\tau_R(z) > \varepsilon_0$.

Again, we will get slightly stronger information on the valid pairs $(\varepsilon_0, \varepsilon_1)$ for Assertion 42, see Section 3.10.

3.9. Step 2, III: Survival of fibers. In this section, we will prove the following proposition (the constant r_0 is defined just before Assertion 42):

Proposition 43. *There exists $K > 0$ and $r'_0 < r_0$ such that for all ε' small enough, for all $f_0 \in \mathcal{F}_0$, for all f_0 -orbit ω_n indexed by $I = \mathbb{Z}$ that tends to 0 in the future (in an attracting petal) and in the past (in a repelling petal), if the orbit (ω_n) is completely contained in $\operatorname{Dom}(f) \odot (1 - \varepsilon')$ then its survival time is at least ε'/K :*

$$\forall n \in \mathbb{Z}, \tau_\Phi(\omega_n) > \varepsilon'/K.$$

Moreover³² there is some $M \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$ with $n \leq M$ and $\forall t \leq \varepsilon'/K$, $\omega_n\langle t \rangle \in D_{\text{rep}}[f_t](r'_0)$.

Here we do *not* need to assume that ε' is related to some $\varepsilon > 0$ like in Proposition 22.

3.9.1. Local orbits. We first consider those orbits that stay near the parabolic point, and prove their survival for some uniform time.

Lemma 44 (Survival of local orbits). *For all $T_3 < T'_1$ there exists $r_1 > 0$ such that for all $f_0 \in \mathcal{F}_0$ and for all f_0 -orbit ω_n indexed by $I = \mathbb{Z}$ or $I = \mathbb{N}$, if the sequence (ω_n) is contained in $B(0, r_1)$, then*

- for all $n \in I$, $\tau_\Phi[f_0](\omega_n) > T_3$,
- if $I = \mathbb{Z}$, then there exists $N \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$ with $n \leq N$ and $\forall t \in [0, T_3]$, $\omega_n\langle t \rangle \in D_{\text{rep}}[f_t](r_0)$.

³²This constant M will of course *not* be independent of the orbit (ω_n) .

Proof. Recall the statements and notations of Propositions 7 and 9 and apply them to the compact set of maps $\mathcal{F}_{[0, T_3]}$, which yields a value r_0 . In their proofs, we introduced the right half plane $H_{\text{attr}}[f]$, image of the disk $D_{\text{attr}}[f]$ by $z \mapsto s(z) = -1/c_f z$. The boundary of H_{attr} is a vertical line of abscissa $1/r_0|c_f|$. Call R_0 the supremum of $1/r_0|c_f|$ when f varies over \mathcal{F}_0 . The function Ψ_{attr} was the inverse of $\Phi_{\text{attr}} : D_{\text{attr}} \rightarrow \Psi_{\text{attr}}(D_{\text{attr}})$. It is important to note a difference: the Fatou coordinates were normalized by their asymptotic expansion in these propositions, whereas here they are normalized using the critical value $v[f_t]$: $\Phi_t(z) = \Phi_{\text{attr}}[f_t](z) + \beta_t$ where $\beta_t = \beta[f_t] = \Phi_{\text{attr}}[f_0](v[f_0]) - \Phi_{\text{attr}}[f_t](v[f_t])$. Let

$$z \mapsto s_t(z) = -1/c[f_t]z.$$

Let $\tilde{\Psi}_t = \Phi_t^{-1}$ defined on $\Phi_t(D_{\text{attr}}[f_t])$. Choose any $T'_3 \in]T_3, T'_1[$. The following three bounds are finite:

$$B = \sup_{f \in \mathcal{F}_{[0, T'_3]}} |c_f|, \quad B' = \sup_{f \in \mathcal{F}_{[0, T'_3]}} |\beta_t| \quad \text{and} \quad \Gamma = \sup_{f \in \mathcal{F}_{[0, T'_3]}} |\gamma[f]|.$$

Since $B' < +\infty$, one can translate the estimates given in Propositions 7 and 9 into estimates on $\tilde{\Psi}_t$ and Φ_t as follows:

$$\begin{aligned} |s_t(f_t(z)) - (s_t(z) + 1)| &\leq 1/4 & (\forall z \in B(0, r_0)) \\ |\Phi_t(s_t^{-1}(u)) - (u - \gamma \log_p u)| &\leq M_1 \\ |s_t \circ \tilde{\Psi}_t(Z) - (Z + \gamma \log_p Z)| &\leq M_2 \\ \text{Dom}(\tilde{\Psi}_t) &\supset \{Z \in \mathbb{C} \mid \text{Re } Z > \xi(\text{Im } Z)\} \\ \xi(y) &\underset{y \rightarrow \pm\infty}{=} \mathcal{O}(\log |y|) \end{aligned}$$

where $s_t, \gamma = \gamma[f_t], \Phi_t$ and $\tilde{\Psi}_t$ all depend on f_t , but the function ξ and the constants M_1, M_2 are independent of f_0 and of t . Consider now a real number $a > R_0$ and the sector $S \subset H_{\text{attr}}$ defined by $\arg(z - a) < \pi/3$. By the first estimate above, $s_t^{-1}(S)$ is stable by f_t . By the other estimates, if a is big enough, for all $f_t \in \mathcal{F}_{[0, T'_3]}$, for all $z \neq 0$, if $s_0(z) \in S$, then $\tilde{\Psi}_t(\Phi_0(z))$ is defined. It follows a fiber of Φ hence by uniqueness in Proposition 34, $\tau_\Phi(z) \geq T'_3$ and

$$z\langle t \rangle = \tilde{\Psi}_t(\Phi_0(z)).$$

Using the estimate above on $\tilde{\Psi}_t$, we get $\forall t \in [0, T'_3], s_t(z)\langle t \rangle \in H_{\text{attr}}[f_t]$ provided $a \geq A'$ for some A' independent of f_0, t and z . Let

$$u(t) := s_t(z\langle t \rangle) = s_t \circ \tilde{\Psi}_t(\Phi_0(z)).$$

In particular $u(0) = s_0(z)$. We then get the following bound on the motion:

$$|u(t) - u(0)| \leq M_4 \log(M'_4 + |u(0)|)$$

where M_4 and M'_4 are independent of t, f_0 and z . Indeed, we start from $|\log_p(x)| \leq \pi + \log |x|$ when $\log |x| > 0$. We then use the estimates above to first get $|\Phi_0(z)| \leq M_1 + |u(0)| + \Gamma\pi + \Gamma \log |u(0)|$ (we can ensure $\log |u(0)| > 0$ by taking $a > 1$) and $|\Phi_0(z)| > 1$ (take a big enough). Then $|u(t)| \leq M_2 + |\Phi_0(z)| + \Gamma\pi + \Gamma \log |\Phi_0(z)| \leq M + M'|u(0)|$ for a pair (M, M') independent of t, f_0, z . Then $|u(t) - u(0)| \leq |u(t) - \Phi_0(z)| + |\Phi_0(z) - u(0)| = |u(t) - \Phi_t(z\langle t \rangle)| + |\Phi_0(z) - u(0)|$ because following a fiber we have $\Phi_t(z\langle t \rangle) = \Phi_0(z)$. Last we use for $t' = t$ and $t' = 0$ that $|u(t') - \Phi_{t'}(z\langle t' \rangle)| \leq M_1 + \Gamma \log |u(t')|$.

So far, we have proved survival of points z with in $s_0(z) \in S$, i.e. $\tau_\Phi(z) \geq T'_3 > T_3$. Figure 28 illustrates the next step of the proof. Let r_1 to be chosen later, with $r_1 < r_0$. Let $R_1 = \inf(1/|c_f r_1|) = 1/r_1 \sup(|c_f|)$ where the extrema are taken over $f \in \mathcal{F}_{[0, T'_3]}$. Assume z_n is an orbit of f_0 indexed by \mathbb{N} that is contained in $B(0, r_1)$.

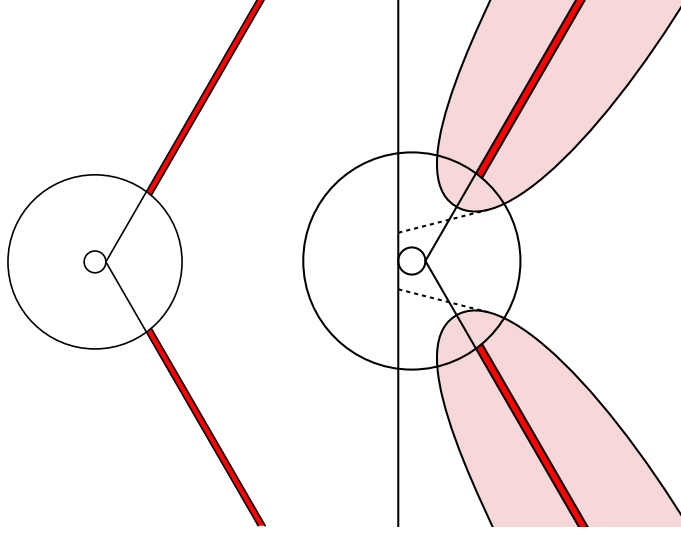


Figure 28: Illustration of the proof of Lemma 44. Both pictures live in the u -plane. The small circle has radius R_0 , the big circle radius R_1 , both are centered on the origin. The sector S has apex sitting nearly on the boundary of the small circle. See the text for further description.

Then the sequence $u_n = s_0(z_n)$ is contained in “ $|u| > R_1$ ”. If $u_0 \in S$ then $\forall n \geq 0$, $\tau_\Phi(z) \geq T'_3$. If $u_0 \notin S$, let n_0 be the smallest positive integer such that $u_{n_0} \in S$ (there is one, by the first estimate in the list). Since $u_{n_0-1} \in “|u| > R_1” \setminus S$ and $u_{n_0} \in “|u| > R_1” \cap S$, the first estimate in the list gives, again, that u_{n_0} must belong to the set Λ , depicted in red in Figure 28, intersection of “ $|u| > R_1$ ” with the set of points in S at distance $\leq 5/4$ from ∂S . By the bound on the motion, $\forall t \in [0, T'_3]$, $u_{n_0}(t)$ belongs to the set Λ' , depicted in light red, union of balls of center $u \in \Lambda$ and of radius $M_4 \log(M'_4 + |u|)$. The map f_t still satisfies the first inequality in the list, hence, provided R_1 is big enough then for all f_0 and for all $t \in [0, T'_3]$, and for all sequence u_n as above, there is an inverse orbit of the conjugate of f_t by s_t , starting from $u_{n_0}(t)$ and remaining in “ $|u| > R_0$ ”, in fact remaining above or below a domain delimited by the dotted line on the figure (on which we interrupted the dotted line when it reaches the repelling petal, delimited by the vertical plain line). By continuity, this orbit is equal to $s_t(z_n(t))$ and $\tau_\Phi(z_n) \geq T'_3 > T_3$, for all $n \in I = \mathbb{Z}$ or \mathbb{N} . If $I = \mathbb{Z}$, let as above n_0 be the smallest relative integer such that $u_{n_0} \in S$. By the first inequality it exists, and moreover the inverse orbit $u_n(t)$, n negative, must enter the repelling petal (and stay there) as soon as $|u_{n_0}| + M_4 \log(M'_4 + |u_{n_0}|) + \frac{3}{4}(n - n_0) < -R_0$. \square

We can in fact bound their motion.

Lemma 45 (Bound on the motion of local orbits). *The following can be added to the conclusions of Lemma 44:*

- $\forall t \in [0, T_3]$, $\forall n \in I$, let $z = \omega_n$: $|z(t) - z| \leq K_1 |z| t$.

The constant K_1 is independent of f_0 , t and z but may depend on T_3 .

Proof. The lazy way uses holomorphic motions:³³ let us extend the deformations f_t to complex values of t in an open neighborhood V of $[0, T_3]$ that does not depend

³³It is possible to avoid holomorphic motions completely, by using Propositions 19, 20 and the remark that follows, which can themselves be proved without holomorphic motions. However, that is much longer.

on $f_0 \in \mathcal{F}$. The hyperbolic length of $[0, T_3]$ in V is thus independent of f_0 . By compactness, this is possible and an analog of Lemma 44 still holds. The function $t \mapsto z\langle t \rangle$ is defined on V and holomorphic.³⁴ Consider the cone of vertex 0, axis \mathbb{R}_+ and angle 3π : this is a Riemann surface over \mathbb{C}^* that is bijectively parameterized in polar coordinates (r, θ) by $]0, +\infty[\times]-\pi/2, \pi/2[$. The study made in the previous lemma shows that, for r_1 small enough, the points ω_n satisfying the assumptions of the theorem have a motion $\omega_n\langle t \rangle$ such that $u_n(t) := -1/c[f_t]\omega_n\langle t \rangle$ stays in this cone when t varies. The element of hyperbolic metric on the cone has expression $c(\theta)|du|/r$ where $c(\theta) \geq c(0) > 0$. The movement of u is holomorphic, hence bounded in this metric by the hyperbolic length of $[0, T_3]$ in V . In Euclidean terms, $u_n(t)$ has moved by at most $Kt|u|$ for some K independent of f_0 . Moreover, $|u_n(t)|$ and $|u_n(0)|$ are of comparable size. Going back to $\omega_n\langle t \rangle = -1/c[f_t]u_n(t)$, we get $|\omega_n\langle t \rangle - \omega_n\langle 0 \rangle| \leq |1/c[f_t]u_n(t) - 1/c[f_t]u_n(0)| + |1/c[f_t]u_n(0) - 1/c[f_0]u_n(0)| \leq |u_n(0) - u_n(t)|/|c[f_t]u_n(0)u_n(t)| + |1/c[f_t] - 1/c[f_0]|/|u_n(0)|$. One concludes recalling $c[f_t]$ is not too close to 0 and depends holomorphically on t . \square

3.9.2. Contraction. Arguments in this section are standard in holomorphic dynamics in complex dimension one.

Let $PC(f_0)$ denote the post critical set of f_0 , i.e. the orbit of the (unique) critical value. Since this orbit tends to 0, the closure $\overline{PC}(f_0)$ equals $PC(f_0) \cup \{0\}$. Let

$$W_0 = \mathbb{C} \setminus \overline{PC}(f_0).$$

It is well known that inverse branches of f_0 are locally contracting for the hyperbolic metric of W_0 . Let us recall the argument: f_0 is a cover from $W'_0 := f_0^{-1}(W_0)$ to W_0 . As such, it is an isometry, at the infinitesimal level, from the hyperbolic metric of W'_0 to that of W_0 . Now $W'_0 \subset W_0$, and strict inclusion maps are locally contracting. Recall that for a hyperbolic domain U of \mathbb{C} we denote $\rho_U(z)|dz|$ the element of hyperbolic metric of U . For $z \in W'_0$, let us denote $\lambda(z)$ the contraction factor of f_0^{-1} from $f_0(z)$ to z , measured with the hyperbolic metric element of W_0 :

$$\lambda(z) = \frac{\rho_{W_0}(z)}{\rho_{W_0}(f_0(z))} \left| \frac{dz}{df_0(z)} \right|;$$

it is also equal to the contraction factor at z of the inclusion map from W'_0 to W_0 :

$$\lambda(z) = \frac{\rho_{W_0}(z)}{\rho_{W'_0}(z)}.$$

The function λ is continuous and takes values in $]0, 1[$.

Let us recall that a hyperbolic open subset of the Riemann sphere with an isolated point a in its complement has a hyperbolic metric coefficient $\rho(z) \sim \frac{1}{2|z-a|\log \frac{1}{|z-a|}}$

as $z \rightarrow a \neq \infty$, or $\rho(z) \sim \frac{1}{2|z|\log |z|}$ as $z \rightarrow a = \infty$.

Lemma 46. *Let $z_n \in W'_0$ be a sequence.*

- (1) *If z_n leaves every compact subset of the open set $W'_0 \cup \overline{PC}(f_0)$, then $\lambda(z_n) \rightarrow 0$.*
- (2) *If $\lambda(z_n) \rightarrow 1$ then $z_n \rightarrow \overline{PC}(f_0)$.*

Proof. We may extract a subsequence and assume z_n convergent in the Riemann sphere.

Point (1): If $z_n \rightarrow \infty$ then $\rho_{W_0}(z) \sim \frac{1}{2|z|\log |z|}$ whereas $\rho_{W'_0}(z) \geq \rho_{\text{Dom}(f_0)}(z)$ and the latter is $\geq \frac{1}{4d_C(z, \partial \text{Dom}(f_0))}$ by Koebe's one quarter theorem. Now since the domain of f_0 is the image of \mathbb{D} by a Schlicht map, there is at least one point in its

³⁴Hence we have a holomorphic motion, because it is injective w.r.t. z , but we will not use that fact.

complement that is at distance at most 1 from 0. Hence $d_{\mathbb{C}}(z, \partial \text{Dom}(f_0)) \leq 1 + |z|$. Putting it all together, we get that $\rho_{W_0}(z)/\rho_{W'_0}(z) \rightarrow 0$ as $|z| \rightarrow +\infty$. In the remaining case: $\lim z_n \neq \infty$ so $\rho_{W_0}(z)$ converges to a constant whereas $\rho_{W'_0}(z) \rightarrow +\infty$.

Point (2): The function λ is continuous and $\lambda(z) < 1$ thus if $\lambda(z_n)$ tends to 1 then z_n leaves every compact subset of W'_0 , and we conclude by the previous point. \square

Lemma 47 (Definite contraction factor at definite distance of PC). *For all $\delta > 0$, there exists $\Lambda(\delta) < 1$ such that $\forall f \in \mathcal{F}$, $\forall z \in W'_0$, if $d_{\mathbb{C}}(z, \overline{PC}(f)) \geq \delta$ then $\lambda(z) \leq \Lambda(\delta)$.*

Proof. If not, there would be sequences $f_n = \mathcal{R}[B_d] \circ \phi_n^{-1} \in \mathcal{F}$ and $z_n \in W'_0[f_n]$ such that $d_{\mathbb{C}}(z_n, \overline{PC}(f_n)) \geq \delta$ but $\lambda[f_n](z_n) \rightarrow 1$. Let us extract convergent subsequences and assume that $z_n \rightarrow z' \in \widehat{\mathbb{C}}$ and $\phi_n \rightarrow \phi$, thus $f_n \rightarrow f = \mathcal{R}[B_d] \circ \phi^{-1}$. Since $\overline{PC}(f)$ is contained in a ball $B(0, R)$ with R independent of $f \in \mathcal{F}$ (Point 2 of Lemma 16), $W_0(f)$ contains $V := \mathbb{C} \setminus \overline{B}(0, R)$, hence $\rho_{W_0}(z) \leq \rho_V(z) \sim 1/2|z| \log |z|$ as $z \rightarrow \infty$. This gives an upper bound like in Point (1) of Lemma 46, but moreover independent of $f \in \mathcal{F}$. It follows that $z' \neq \infty$. By Lemma 16, $\overline{PC}(f)$ depends continuously on ϕ thus $d_{\mathbb{C}}(z', \overline{PC}(f)) \geq \delta$. Hence $z' \in W_0[f]$. Now the marked domains $(W_0[f_n], z_n)$ converge for the Caratheodory topology on marked domains. Hence their universal cover from $(\mathbb{D}, 0)$ with real positive derivative at the origin converge, and the coefficient of the hyperbolic metric converges locally uniformly: $\rho_{W_0[f_n]}(z_n) \rightarrow \rho_{W_0[f]}(z')$. Concerning the marked domains $(W'_0[f_n], z_n)$, there are two cases: either $z' \in W'_0[f]$ in which case there is Caratheodory convergence to $(W'_0[f], z')$ and thus $\rho_{W'_0[f_n]}(z_n) \rightarrow \rho_{W'_0[f]}(z')$; or $z' \notin W'_0[f]$ in which case³⁵ $\rho_{W'_0[f_n]}(z_n) \rightarrow +\infty$. In the first case $\lambda[f_n](z_n) \rightarrow \lambda[f](z') < 1$. In the second case $\lambda[f_n](z_n) \rightarrow 0$. Both cases lead to a contradiction. \square

3.9.3. *Putting back the post critical set.* The following easy lemma will be useful in several places.

Lemma 48. *There exists a function $\delta > 0 \mapsto M(\delta) > 0$ such that the following holds. For all $f \in \mathcal{F}$, for all $z \in \text{Dom}(f)$, if $d_{\mathbb{C}}(z, PC(f)) \geq \delta$ then*

$$\frac{\rho_{W_0(f)}(z)}{\rho_{\text{Dom}(f)}(z)} \leq M(\delta).$$

Proof. In this proof, the notation $B(z, r)$ denotes the euclidean ball and $PC = PC(f)$. By Lemma 16, there is $R > 0$ such that for all $f \in \mathcal{F}$, $PC \subset \overline{B}(0, R)$. Let $U = \mathbb{C} \setminus \overline{B}(0, R)$. Then

$$\rho_{W_0}(z) \leq \rho_U(z) = \frac{1}{2|z| \log \frac{|z|}{R}}.$$

Since the disk D of center z and radius $d_{\mathbb{C}}(z, PC)$ is contained in W_0 , we get

$$\rho_{W_0}(z) \leq \rho_D(z) = \frac{1}{d_{\mathbb{C}}(z, PC)}.$$

³⁵Indeed, there exists then a point $x_n \in \mathbb{C} \setminus W'_0[f_n]$ such that $x_n \rightarrow z'$. Let $r' = |z'|$ and let $r'' \geq 1$ be any real such that $r'' \neq r'$, for instance $r'' = r' + 1$. Since the conformal radius w.r.t. 0 of the simply connected set $\text{Dom } f_n$ is 1, there exists a point in $\mathbb{C} \setminus \text{Dom } f_n$ of any modulus ≥ 1 , in particular a point y_n of modulus r'' . Let $V_n = \mathbb{C} \setminus \{x_n, y_n\}$. Then $\rho_{W'_0[f_n]}(z_n) \geq \rho_{V_n}(z_n)$. Let ϕ_n be the unique \mathbb{C} -affine map sending 0 to x_n and 1 to y_n and let $u_n = \phi_n^{-1}(z_n)$. Then $\phi'_n = y_n - x_n$ and $\rho_{\mathbb{C} \setminus \{0, 1\}} = \phi_n^*(\rho_{V_n}) = |\phi'_n| \times \rho_{V_n} \circ \phi_n$. For n big enough, the sequence $x_n - y_n$ is bounded away from 0 (and ∞) thus $u_n \rightarrow 0$ thus $\rho_{\mathbb{C} \setminus \{0, 1\}}(u_n) \rightarrow +\infty$ and also $\rho_{V_n}(z_n) = \rho_{\mathbb{C} \setminus \{0, 1\}}(u_n)/|y_n - x_n| \rightarrow +\infty$.

By the theory of univalent functions,

$$\rho_{\text{Dom}(f)}(z) \geq \frac{1}{4(1+|z|)}.$$

The lemma follows. \square

3.9.4. Homotopic length and decomposition. For γ a path defined on an interval I containing $[a, b]$, let us denote its restriction to $[a, b]$ by

$$\gamma|_{[a, b]}.$$

Let us similarly denote

$$\omega_n\langle[0, t]\rangle : s \in [0, t] \mapsto \omega_n\langle s\rangle.$$

To bound the motion of $\omega_n\langle t\rangle$ we will look at the *homotopic length* of the path $\omega_n\langle[0, t]\rangle$ for the hyperbolic metric on $W_0 = \mathbb{C} \setminus \overline{PC}(f_0)$. Homotopic length of a path γ refers to the infimum of W_0 -hyperbolic lengths of paths homotopic to γ in W_0 , where the ends of the path are fixed. It will be denoted

$$\text{hlen}_{W_0}(\gamma).$$

By contrast, we denote as follows the usual length of a rectifiable path for the hyperbolic metric of W_0 :

$$\text{len}_{W_0}(\gamma).$$

Last, we will call *extent* of a path γ defined on $[0, t]$ the quantity

$$\text{extent}_{W_0}(\gamma) = \sup_{t' \in [0, t]} \text{hlen}_{W_0}(\gamma|_{[0, t']}).$$

Remark. Homotopic length is also the hyperbolic distance between the starting point and the end point of a lift of the curve to the universal cover. There are in particular shortest homotopic paths. The extent of a curve is the smallest radius of a ball in the universal cover containing a lift of the curve and centered on the initial point of this lift. If U is connected and $\gamma \subset U \subsetneq V$ then the V -homotopic length of γ is strictly smaller than its U -homotopic length: consider for instance the shortest homotopic path for V ; its U -length is strictly shorter. If U and V are hyperbolic Riemann surfaces and $f : U \rightarrow V$ is a cover then $\text{hlen}_U(\gamma) = \text{hlen}_V(f \circ \gamma)$.

Remark. The sequence $(\omega_n\langle t\rangle)_{n \in \mathbb{N}}$ is an orbit of f_t , not f_0 . It may therefore seem unnatural to measure the motion of $t \mapsto \omega_n\langle t\rangle$ using the hyperbolic metric on the complement of $\overline{PC}(f_0)$. However, we found the proof simpler to write that way. Note that the motion will be evaluated only at some distance from the post critical points, and in the end it will be small.

Recall that $f_0 \in \mathcal{F}$ decomposes as

$$f_0 = \mathcal{R}[B_d] \circ \phi_0^{-1}$$

with $\phi_0 : \mathbb{D} \rightarrow U_0$ a Schlicht map. Let us decompose the map f_t as follows:

$$f_t = f_0 \circ \sigma_t$$

where

$$\sigma_t(z) = \phi_0 \circ r_t \circ \phi_0^{-1} \circ r_t^{-1}$$

with the notations of Section 3.7.1 and letting r_t denote the multiplication by

$$r_t = 1 - t.$$

The map σ_0 is the identity restricted to $\text{Dom } f_0$. If we interpret $\sigma_t(z)$ as a motion of z as t varies, then it can be viewed as the composition of two motions: $(t, z) \mapsto$

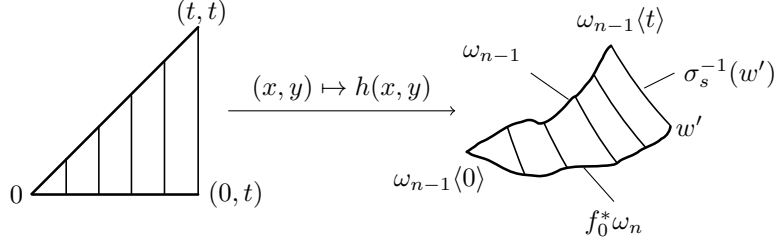


Figure 29: The map $h(x, y) = \sigma_y^{-1}(f_0^* \omega_n(x))$ defined on the triangle “ $x \in [0, t]$, $y \in [0, t]$, $y \leq x$ ” induces a homotopy between ω_{n-1} on $[0, t]$ and the concatenation of $f_0^* \omega_n$ and $s \in [0, t] \mapsto \sigma_s^{-1}(w')$.

$(t, r_t^{-1}z)$ followed by the conjugate by ϕ_0 of the radial motion $(t, z) \mapsto (t, r_t z)$ on the unit disk:

$$\sigma_t = \mu_t \circ r_t^{-1}$$

with

$$\mu_t = \phi_0 \circ r_t \circ \phi_0^{-1}.$$

The domain of definition of the reciprocal σ_t^{-1} equals $\phi_0(B(0, r_t)) = \text{Dom}(f_0) \odot r_t$ and thus as t varies away from 0, it shrinks.

Now recall that the path $s \mapsto \omega_{n-1}\langle s \rangle$ is defined inductively by continuity via $f_s(\omega_{n-1}\langle s \rangle) = \omega_n\langle s \rangle$, for s as big as possible. Consider the case where $\omega_n\langle 0 \rangle$ is not equal to 0 nor to the singular value v of f_0 . Then $\omega_n\langle s \rangle \notin \{0, v\}$, because 0 and v do not move under the fiberwise motion, and Φ -fibers are disjoint. Recall that the singular values of f_0 are precisely 0, ∞ and v . We claim that, under some condition stated below, the path $s \in [0, t] \mapsto \omega_{n-1}\langle s \rangle$ is homotopic (with endpoints fixed) in W_0 to the concatenation of the following two paths (see Figure 29):

- The first path, denoted $\gamma_1 = f_0^* \omega_n$ by a slight abuse of notation, is parameterised by $s \in [0, t]$ and is defined by continuity by $\gamma_1(0) = \omega_{n-1}\langle 0 \rangle$ and $f_0(\gamma_1(s)) = \omega_n\langle s \rangle$, i.e. we replaced f_s by f_0 in $f_s(\omega_{n-1}\langle s \rangle) = \omega_n\langle s \rangle$. Existence of this path follows from $\omega_n\langle s \rangle$ never hitting the singular values of f_0 . It ends at some point w' (which depends on t);
- The second path is $\gamma_2 : s \in [0, t] \mapsto \sigma_s^{-1}(w')$. For it to be defined up to $s = t$, we need to assume that $w' \in \text{Dom}(f_0) \odot (1 - t) = \phi_0(B(0, 1 - t))$.

The homotopy will be defined by means of a map h defined on the set of $(x, y) \in [0, t]^2$ such that $y \leq x$ by

$$h(x, y) = \sigma_y^{-1}(\gamma_1(x)).$$

For it to be well defined, we will make assumptions on t , on the length of ω_n and on the ε such that $\omega_{n-1}\langle 0 \rangle \in \text{Dom}(f_0) \odot (1 - \varepsilon)$. For it to be a homotopy in W_0 , we need to prove that its support does not intersect $\overline{PC}(f_0)$ and for this we will make further assumptions on t , on the length of ω_n and on the Euclidean distance from $\omega_{n-1}\langle 0 \rangle$ to $\overline{PC}(f_0)$.

To state these sufficient conditions, we will introduce the following objects and quantities. For $\delta > 0$ let $V_\delta[f]$ denote the δ -neighborhood of $PC(f)$, i.e. the set of points whose Euclidean distance to $PC(f)$ is $< \delta$ (see Figure 30). According to Lemma 16, the following quantity is positive:

$$\delta_1 := \inf_{f_0 \in \mathcal{F}_0} d_{\mathbb{C}}(PC(f_0), \mathbb{C} \setminus \text{Dom } f_0)$$

where $d_{\mathbb{C}}$ refers to the Euclidean distance, and the following are finite:

$$R_1 := \sup \{|z| \mid z \in PC(f_0), f_0 \in \mathcal{F}_0\},$$

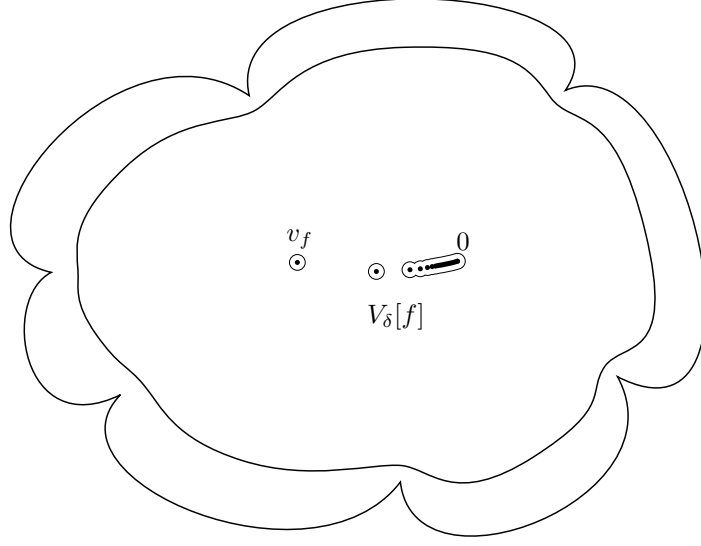


Figure 30: A schematic illustration of $\text{Dom}(f)$, $\text{Dom}(f) \odot r$ and $V_\delta[f]$. Scales are not respected. The outer curve represents the boundary of the domain of some $f \in \mathcal{F}$, the nearby smooth curve the boundary of the sub-domain $\text{Dom}(f) \odot 1 - \varepsilon$. The post-critical set is indicated by dots, its δ -neighborhood for the Euclidean metric is $V_\delta[f]$ and its boundary is indicated by thin curves.

$$R_2 := \sup \{ d_{\text{Dom } f_0}(0, z) \mid z \in PC(f_0), f_0 \in \mathcal{F}_0 \}.$$

Lemma 49. *For all (δ, δ') with $\delta' < \delta < \delta_1$, there exists $T = T(\delta, \delta') > 0$ such that $\forall f_0 \in \mathcal{F}_0, \forall t < T$:*

- $\mu_t^{-1}(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0] = \emptyset$,
- $r_t(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0] = \emptyset$,
- $\sigma_t^{-1}(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0] = \emptyset$.

Proof. We can deduce the third point from the first two, using an intermediary value δ'' . This may not be optimal³⁶ but it is not the point here. For the second point, an explicit valid value of T can easily be computed using Lemma 16: assume $z \in r_t(\mathbb{C} \setminus V_\delta[f_0]) \cap V_{\delta'}[f_0]$. Then there exists $z' \in PC(f_0)$ such that $|z - z'| < \delta'$, thus $|z| < R_1 + \delta'$. Then $|z - r_t^{-1}z| \leq (R_1 + \delta')(r_t^{-1} - 1)$. If T is chosen so that $(R_1 + \delta')(r_T^{-1} - 1) < \delta - \delta'$ then $r_t^{-1}z$ cannot belong to $\mathbb{C} \setminus V_\delta[f_0]$. For the first point, let us work by contradiction and assume there is $f_n \in \mathcal{F}_0, a_n \in \mathbb{C} \setminus V_\delta[f_n], b_n \in V_{\delta'}[f_n]$ and $t_n > 0$ such that $t_n \rightarrow 0$ and $a_n = \mu_{t_n}(b_n)$. We may assume that $f_n \rightarrow f \in \mathcal{F}_0, a_n \rightarrow a \in \widehat{\mathbb{C}}$ and $b_n \rightarrow b \in \mathbb{C}$. From $|a_n - b_n| > \delta - \delta'$ we get $|a - b| \geq \delta - \delta'$. Write $f_n = \mathcal{R}[B_d] \circ \phi_n^{-1}$ and $f = \mathcal{R}[B_d] \circ \phi^{-1}$. From $\delta' < \delta_1$ and $R_2 < +\infty$ we deduce that $\phi_n^{-1}(b_n)$ remains in a compact subset of \mathbb{D} thus b belongs to $\text{Dom}(f)$, but then $a = \mu_0(b) = b$, a contradiction. \square

We will later choose some

$$\delta < \delta_1.$$

³⁶Near $z = 0$, the Euclidean motion of σ_t is of order $|z|^2$, thus smaller than the sum of the motions of μ_t and of r_t , which are both of order $|z|$.

Let then

$$d_1 = d_1(\delta) = \inf_{f_0 \in \mathcal{F}_0} d_{W'_0}(V_{\delta/3}[f_0], \mathbb{C} \setminus V_{\delta/2}[f_0])$$

v being the critical value of f_0 . Let also

$$d''_1 = d''_1(\delta) = \inf_{f_0 \in \mathcal{F}_0} d_{f^{-1}(\mathbb{C} \setminus \{0, v\})}(V_{\delta/3}[f_0], \mathbb{C} \setminus V_{\delta/2}[f_0])$$

and note that $d''_1 < d_1$. Using the notation of Lemma 49 let

$$T_4(\delta) = T(\delta/3, \delta/4)$$

so that $\forall f_0 \in \mathcal{F}_0, \forall t < T_4(\delta), \sigma_t^{-1}(\mathbb{C} \setminus V_{\delta/3}[f_0]) \cap V_{\delta/4}[f_0] = \emptyset$. Let $\ell(x)$ denote the hyperbolic distance from 0 to x in \mathbb{D} :

$$\ell(x) = d_{\mathbb{D}}(0, x) = \operatorname{argth}(x).$$

It is a bijection from $[0, 1[$ to $[0, +\infty[$. For a given $\varepsilon' > 0$, let $T_5 = T_5(\delta, \varepsilon') \in]0, 1[$ be the unique solution to

$$\ell(1 - T_5) = d_1(\delta) + \ell(1 - \varepsilon').$$

Note that the solution T''_5 of $\ell(1 - T''_5) = d''_1(\delta) + \ell(1 - \varepsilon')$ satisfies $T''_5 > T_5$. We will later look at how $T_5(\delta, \varepsilon')$ varies as $\varepsilon' \rightarrow 0$ for a fixed δ . Recall the definition of extent given at the beginning of the present section on page 72.

Proposition 50. *Let $t > 0$. If we assume that*

- (1) $\tau_{\Phi}(\omega_n\langle 0 \rangle) > t$,
- (2) *the path $s \in [0, t] \mapsto \omega_n\langle s \rangle$ is contained in W_0 ,*
- (3) $\operatorname{extent}_{W_0}(\omega_n\langle [0, t] \rangle) \leq d_1(\delta)$,
- (4) $\omega_{n-1}\langle 0 \rangle \in \operatorname{Dom}(f_0) \odot (1 - \varepsilon')$,
- (5) $\omega_{n-1}\langle 0 \rangle \notin V_{\delta/2}[f_0]$,
- (6) $t \leq T_4(\delta)$,
- (7) $t \leq T_5(\delta, \varepsilon')$,

then $\tau_{\Phi}(\omega_{n-1}\langle 0 \rangle) > t$ and the function h mentioned above is well defined and has support in W_0 (even better: it avoids $V_{\delta/4}[f]$). In particular $s \in [0, t] \mapsto \omega_{n-1}\langle s \rangle$ is homotopic in W_0 to the concatenation $\gamma_1 \cdot \gamma_2$, of the two paths defined earlier, page 73.

Proof. By (2) the path ω_n is contained in $\mathbb{C} \setminus \{0, v\}$ thus the path γ_1 , defined as the pull-back by f_0 of $\omega_n\langle \cdot \rangle$ starting from $\omega_{n-1}\langle 0 \rangle$, is well defined. Let $t' \in [0, t]$:

$$\operatorname{hlen}_{\operatorname{Dom} f_0}(\gamma_1|_{[0, t']}) < \operatorname{hlen}_{W'_0}(\gamma_1|_{[0, t']}) = \operatorname{hlen}_{W_0}(\omega_n\langle \cdot \rangle|_{[0, t']}) \leq d_1$$

(the first inequality comes from the strict inclusion $W'_0 \subset \operatorname{Dom} f_0$, the equality follows from f_0 being a cover from W'_0 to W_0 , the second inequality comes from point (3)). In particular the $\operatorname{Dom} f_0$ -hyperbolic distance from $\gamma_1(0)$ to $\gamma_1(t')$ is $\leq d_1$. Since moreover by (4), $d_{\operatorname{Dom} f_0}(0, \omega_{n-1}\langle 0 \rangle) \leq \ell(1 - \varepsilon')$ we get that γ_1 is contained in the $\operatorname{Dom} f_0$ hyperbolic ball of center 0 and radius $d_1 + \ell(1 - \varepsilon') = \ell(1 - T_5)$. Hence $\gamma_1 \subset \operatorname{Dom}(f_0) \odot (1 - T_5)$. Hence by (7), γ_2 and the map h defined at the same place are well defined. Let us check that h takes values in W_0 , i.e. that it avoids $\overline{PC}(f_0)$. Note that we have already proved that $\operatorname{hlen}_{W'_0}(\gamma_1|_{[0, t']}) \leq d_1$. In particular the W'_0 -hyperbolic distance from $\gamma_1(0)$ to $\gamma_1(t')$ is $\leq d_1$. Together with point (5) and the definition of d_1 , it implies that γ_1 is contained in $\mathbb{C} \setminus V_{\delta/3}[f_0]$. Point (6) then implies that γ_2 and h take value in $\mathbb{C} \setminus V_{\delta/4}[f_0]$, which is contained in W_0 . The points $h(s, s)$ and $\omega_{n-1}\langle s \rangle$ are both mapped by f_s to the same point: $\omega_n\langle s \rangle$. A continuity argument on s implies that they are in fact equal and that $\tau_{\Phi}(\omega_{n-1}) > t$. \square

We have the following variation with W_0 replaced by $\mathbb{C} \setminus \{0, v\}$ in the hypotheses, but not in the conclusion:

Proposition 51. *Let $t > 0$. If we assume that*

- (1) $\tau_\Phi(\omega_n\langle 0 \rangle) > t$,
- (2) *the path $s \in [0, t] \mapsto \omega_n\langle s \rangle$ is contained in $\mathbb{C} \setminus \{0, v\}$,*
- (3) $\text{extent}_{\mathbb{C} \setminus \{0, v\}}(\omega_n\langle [0, t] \rangle) \leq d_1''(\delta)$,
- (4) $\omega_{n-1}\langle 0 \rangle \in \text{Dom}(f_0) \odot (1 - \varepsilon')$,
- (5) $\omega_{n-1}\langle 0 \rangle \notin V_{\delta/2}[f_0]$,
- (6) $t \leq T_4(\delta)$,
- (7) $t \leq T_5''(\delta, \varepsilon')$,

then $\tau_\Phi(\omega_{n-1}\langle 0 \rangle) > t$ and the function h is well defined and avoids $V_{\delta/4}[f]$. In particular it has support in W_0 and the path $s \in [0, t] \mapsto \omega_{n-1}(s)$ is homotopic in W_0 to $\gamma_1 \cdot \gamma_2$.

Proof. As in the previous proof. \square

Lemma 52. *Under the conditions of Proposition 50, the W_0 -homotopic length of γ_1 is at most $\Lambda(\delta/3)$ times the W_0 -homotopic length of ω_n , where by $\Lambda(\delta/3) < 1$ is given by Lemma 47.*

Proof. We have seen that $\text{hlen}_{W'_0}(\gamma_1) \leq d_1$. Consider a shortest path γ homotopic to γ_1 in W'_0 : $\text{len}_{W'_0}(\gamma) = \text{hlen}_{W'_0}(\gamma_1)$. It is a geodesic for the hyperbolic metric, in particular all its points are at W'_0 -hyperbolic distance $\leq d_1$ from its starting point. By the definition of d_1 this implies that γ is disjoint from $V_{\delta/3}[f_0]$. By Lemma 47, we have $\lambda(z) \leq \Lambda(\delta/3)$ for z in the support of γ , with $\lambda(z) = \rho_{W_0}(z)/\rho_{W'_0}(z)$. Thus $\text{hlen}_{W_0}(\gamma_1) \leq \text{len}_{W_0}(\gamma) \leq \Lambda(\delta/3) \text{len}_{W'_0}(\gamma) = \Lambda(\delta/3) \text{hlen}_{W_0}(\omega_n)$. \square

Lemma 53. *Under the conditions of Proposition 51, the W_0 -homotopic length of γ_1 is at most $M(\delta/3)$ times the $\mathbb{C} \setminus \{0, v\}$ -homotopic length of ω_n , where $M(\dots)$ is given in Lemma 48.*

Proof. This done as in the previous lemma, with W'_0 replaced by $f^{-1}(\mathbb{C} \setminus \{0, v\})$, d_1 by d_1'' and $\lambda(z)$ by $\rho_{W_0}(z)/\rho_{f^{-1}(\mathbb{C} \setminus \{0, v\})}(z)$. By inclusion, the latter quantity is $\leq \rho_{W_0}(z)/\rho_{\text{Dom } f}(z)$ thus $\leq M(\delta/3)$. \square

The W_0 -homotopic length of γ_2 will be controlled using Lemma 54 below. To state it we need to introduce another quantity. By Lemma 49 there exists $T_6 = T_6(\delta)$ such that $\forall f_0 \in \mathcal{F}_0, \forall t < T_6, \mu_t^{-1}(\mathbb{C} \setminus V_{\delta/4}[f_0]) \cap V_{\delta/5}[f_0] = \emptyset$ and $r_t(\mathbb{C} \setminus V_{\delta/5}[f_0]) \cap V_{\delta/6}[f_0] = \emptyset$.

Lemma 54. *For all $\delta < \delta_1$, there exists $K_0 = K_0(\delta)$ such that under the conditions of Proposition 50 or 51, and assuming moreover*

- $t \leq T_6(\delta)$ and $t \leq T_5(\delta, \varepsilon')/2$

then the W_0 -homotopic length of γ_2 is $\leq K_0 t / T_5(\delta, \varepsilon')$.

Proof. Similarly to the proof of the propositions, the condition $t < T_6$ ensures that there is a homotopy in W_0 between γ_2 and $\gamma_3 \cdot \gamma_4$ where $\gamma_3(s) = \mu_s^{-1}(\gamma_2(0))$ and $\gamma_4(s) = r_s w''$ where w'' is the endpoint of γ_3 . The motion μ_s is the conjugate by ϕ_0 of the radial motion and we have seen that $x := |\phi_0^{-1}(\gamma_2(0))| \leq 1 - T_5$ (in the case of Proposition 51 we have $x \leq 1 - T_5'' < 1 - T_5$) thus the length of γ_3 for the hyperbolic metric of $\phi_0(\mathbb{D}) = \text{Dom } f_0$ is $\leq d_{\mathbb{D}}(x, \frac{x}{1-t}) \leq d_{\mathbb{D}}(1 - T_5, \frac{1-T_5}{1-t}) = \frac{1}{2} \log \left(\frac{1 - \frac{1-T_5}{1-t}}{1 - \frac{1-T_5}{1-t}} \right) \leq \frac{1}{2} \times -\log(1 - \frac{t}{1-T_5}) \leq t \log(2)/T_5$, the latter because $t/T_5 \leq 1/2$. Lemma 48 implies that its W_0 -length is at most $M(\delta/5)$ times this quantity. To bound the W_0 -length of γ_4 , note that it is contained in the complement of $V_{\delta/6}$, thus $\forall z \in \gamma_4, \rho_{W_0}(z) \leq 6/\delta$. Also, $\rho_{W_0}(z) \leq 1/(|z| - R_1)$ where $R_1 = \sup \{|z| \mid f \in \mathcal{F}, z \in PC(f)\}$. If $|w''| > 4R_1$ then since $t \leq 1/2$, the whole path γ_4 is contained in the complement

of $B(0, 2R_1)$ and thus $\rho_{W_0} \leq 2/|z|$ whence a W_0 -length of γ_4 that is $\leq \int_{|z|}^{\frac{|z|}{1-t}} \frac{2}{x} dx = 2 \log(1/(1-t)) \leq 4t \log 2$ because $t \leq 1/2$. If $|w''| \leq 4R_1$ then the whole euclidean length of γ_4 is $\leq 4R_1 t$ hence the W_0 -length is $\leq (6/\delta)4R_1 t$. \square

Remark. The linearity of the bound w.r.t. t is not crucial for this article: weaker orders of convergence to 0 would work for our purpose, thanks to the fact that in Proposition 22, ε' is much bigger than ε . What will be important is that values of t for which the bound is a given small constant are much bigger than ε . So how T_5 depends on ε will be important too (recall δ will be fixed).

Remark. Lemmas 52, 53 and 54 only give an upper bound on the W_0 -homotopic length of the curve $s \in [0, t] \mapsto \omega_{n-1}\langle s \rangle$ but on each of its subsegments $s \in [0, t']$ for $t' < t$, by applying Proposition 50 to t' instead of t . So we get in fact bounds on the extent. This allows for induction.

3.9.5. Visits in the repelling petal.

Lemma 55. *There exists $K_4 > 0$ such that $\forall f \in \mathcal{F}$, $\forall z \in W_0$, if $|z| \leq 1$ then $\rho_{W_0}(z) \geq 1/K_4|z|$.*

Proof. Let us work by contradiction and assume that there exist $f_n \in \mathcal{F}$ and $z_n \in \mathbb{D}$ such that $z_n \in W_0[f_n]$ and $\rho_{W_0[f_n]}(z_n)|z_n| \rightarrow 0$. Consider the dilatation by $1/z_n$: the set $z_n^{-1}W_0[f_n] \subset \mathbb{C}$ does not contain 0, but it contains the point 1 and has a hyperbolic metric coefficient at this point tending to 0 as $n \rightarrow +\infty$. Since \mathbb{C} minus two points is hyperbolic, and since inclusion is non-expanding for the hyperbolic metric, there would thus exist $R_n, r_n > 0$ such that $R_n > |z_n| > r_n$, such that $W_0[f_n]$ contains the annulus “ $r_n < |z| < R_n$ ” and such that $R_n/|z_n| \rightarrow +\infty$ and $r_n/|z_n| \rightarrow 0$.

Let us apply Lemma 14 to $r = r_0$ where r_0 is provided by Proposition 7. The point $f^{n_0}(v_f)$ belongs to $D_{r_0}[f]$. It depends continuously on f and thus it remains in a compact subset of $\mathbb{C} \setminus \{0\}$. Let $a_n[f] = -1/c_f f^n(v_f)$ where c_f is defined in Proposition 7 and is bounded away from 0 and ∞ as f varies in \mathcal{F} . Then $a_{n_0}[f]$ also belongs to a compact set. Hence $\forall n \geq 0$, $3n/4 - A \leq |a_{n_0+n}[f]| \leq A + 5n/4$ for a constant $A > 0$ that is independent of f . It follows that there is $A', A'' > 0$ and $n_1 \geq n_0$ such that for all $f \in \mathcal{F}$ and for all $n \geq n_1$, $\frac{A'}{n} \leq |f^n(v_f)| \leq \frac{A''}{n}$.

Hence the aforementioned sequence of annuli cannot exist, which yields a contradiction. \square

In coordinates $u = -1/c_f z$ this reads $\rho_{-1/(c_f W_0)}(u) \geq 1/K_4|u|$.

Proposition 56. *There exists r_2, T_8 and d'_1 , positive reals, such that for all $f_0 \in \mathcal{F}$, for all $n_0, n_1 \in \mathbb{Z}$ with $n_0 < n_1$, for all f_0 -orbit ω_n indexed by $\mathbb{Z} \cap [n_0, +\infty[$, and for all $t > 0$, if*

- (1) $\omega_{n_0}\langle 0 \rangle, \dots, \omega_{n_1}\langle 0 \rangle \in D_{\text{rep}}[f_0](r_2)$,
- (2) $\tau(\omega_{n_1}) > t$,
- (3) $t \leq T_8$,
- (4) $\text{extent}_{W_0}(\omega_{n_1}\langle [0, t] \rangle) < d'_1$,

then $\tau(\omega_{n_0}) > t$, the paths γ_1 and γ_2 defined below are well defined, and $s \in [0, t] \mapsto \omega_{n_0}\langle s \rangle$ is homotopic in $W_0[f_0]$ to the concatenation $\gamma_1 \cdot \gamma_2$. The path γ_1 is the pull-back of $s \in [0, t] \mapsto \omega_{n_1}\langle s \rangle$ by $f_0^{n_1-n_0}$ that starts from $\omega_{n_0}\langle 0 \rangle$; the path $\gamma_2 : [0, t] \rightarrow \mathbb{C}$ is the continuous solution, starting from $\gamma_1(t)$, of $f_s^{n_1-n_0}(\gamma_2(s)) = \text{const} = f_0^{n_1-n_0}(\gamma_1(t)) = \omega_{n_1}\langle t \rangle$.

Proof. Consider the domains $W_\theta(R)$ and $D_\theta(r_0)[g]$ introduced in Section 3.5.2, with $-1/c_g D_\theta(r_0) = W_\theta(1/|c_g|r_0)$. We will take some $T_8 \leq 1/2$. The class of maps $\mathcal{F}_{[0,1/2]}$ is compact and the domain of its members all contain $B(0, 1/8)$, so we can apply Propositions 18 and 21 to the restriction to \mathbb{D} of the conjugates of maps in this class by $z \mapsto 8z$. Choose $\theta = 3\pi/4$, $\theta' = (\theta + \frac{\pi}{2})/2$, $\theta'' = (\theta' + \frac{\pi}{2})/2$, so that $\pi/2 < \theta'' < \theta' < \theta$. It was proved in Proposition 18 that for $r_0 > 0$ small enough, the (invertible) repelling Fatou coordinates of $g \in \mathcal{F}_{[0,1/2]}$ extend to $-D_\theta(r_0)[g]$ for some $r_0 > 0$, and that $-D_\theta(r_0)[g]$, $-D_{\theta'}(r_0)[g]$ and $-D_{\theta''}(r_0)[g]$ are all invariant by a branch of g^{-1} . By compactness, for r_0 small enough, there is only one such branch. Also, provided r_0 has been chosen small enough, it can be checked using Proposition 7 and Lemma 14 that $PC[g]$ does not intersect $-D_\theta(r_0)[g]$.

Now choose any $r_1 < r_0$, for instance $r_1 = r_0/2$ and impose $r_2 \leq r_1$. Let $\gamma_0 : [0, t] \rightarrow \mathbb{C}$, $s \mapsto \omega_{n_1}\langle s \rangle$: by assumption its initial point is contained in $D_{\pi/2}[f_0](r_2)$. By Lemma 55, for d'_1 small enough, we are ensured that γ_0 is contained in $-D_{\theta''}(r_0)[f_0]$ (this is more easily seen in coordinates $u = -1/c_{f_0}z$: the path stays in a ball of center its initial point u_0 and radius $\mathcal{O}(d'_1|u_0|)$). Since $-D_{\theta''}(r_0)[f_0]$ is stable by a branch of f_0^{-1} , the path γ_1 is well defined and contained in $-D_{\theta''}(r_0)[f_0]$. Now, as in the proof of Proposition 50, we set up a triangular homotopy $h(x, y)$ for $y \leq x \leq t$ with $f_y^{n_1-n_0}(h(x, y)) = f_0^{n_1-n_0}(\gamma_1(x)) = \gamma_0(x)$ and $h(x, 0) = \gamma_1(x)$. Taking T_8 small enough, we get $-D_{\theta''}(r_0)[f_0] \subset -D_{\theta'}(r_0)[f_y]$ for all $y \leq T_8$ and all $f_0 \in \mathcal{F}$. In particular $\gamma_0 \subset -D_{\theta'}(r_0)[f_y]$. Since the latter is invariant by a branch of f_y^{-1} , unique and continuously depending on y , it follows that h is well-defined, continuous, and has support in $-D_{\theta'}(r_0)[f_y]$. For T_8 small enough, $-D_{\theta'}(r_0)[f_y] \subset -D_\theta(r_0)[f_0]$, hence h takes values in $W_0[f_0]$. \square

This proof yields more:

Complement. *There is some $r_4 > 0$ and $\theta' > 0$ such that under the conditions of the proposition above, and $\forall s \in [0, t]$, $-D_{\theta'}[f_s](r_4)$ is a repelling petal for f_s and for all k with $n_0 \leq k \leq n_1$, $\omega_k\langle s \rangle \in -D_{\theta'}[f_s](r_4)$.*

Proof. Change the value of n_0 to that of k in the previous proposition. Its proof provided some quantities called r_0 and θ' , and proved the claim of the complement for $r_4 = r_0$ and the same value of θ' . \square

Note that by infinitesimal contraction of f_0^{-1} for the hyperbolic metric of W_0 ,

$$\text{hlen}_{W_0}(\gamma_1) < \text{hlen}_{W_0}(s \in [0, t] \mapsto \omega_{n_0}\langle s \rangle).$$

Since γ_2 stays far from the boundary of W_0 , the control we get on its homotopic length is better than in Lemma 54:

Lemma 57. *We can add the following conclusion to the previous lemma*

$$\text{hlen}_{W_0}(\gamma_2) \leq K_5 t.$$

Proof. In this proof we will say that a constant is independent if it is independent of f , of the chosen orbit ω_n and of the length $n_1 - n_0$. We will use $\mathcal{O}(\text{expression})$ to express a quantity that is at most the expression times a constant that is independent. We will write that two quantities are comparable when their quotient is bounded away from 0 and ∞ independently of f , of the chosen orbit ω_n and of the length $n_1 - n_0$. Let us continue with the notations of the previous proof. Note that $\gamma_2(y) = h(t, y)$ and $\gamma_2(t) \in -D_\theta(r_0)[f_0]$. Since there are sectors $-D_{\theta_3}(r_3)[f_0]$ contained in $W_0[f_0]$ for $\theta_3 = (\theta + \pi)/2 > \theta$ with r_3 independent of f_0 , by imposing $r_0 < r_3$, we have $\forall z \in -D_\theta(r_0)[f_0]$, $B(z, |z|/K) \subset W_0[f_0]$. Hence it is enough to prove that for $y \leq t$,

$$|\gamma_2(y) - \gamma_2(0)| = \mathcal{O}(K' t |\gamma_2(0)|),$$

in which case, for $t < T_8$ with T_8 small enough, the euclidean ball $B(\gamma_2(0), K't|\gamma_2(0)|)$ is contained in $W_0[f_0]$ and contains γ_2 thus γ_2 is homotopic in $W_0[f_0]$ to the straight segment from $\gamma_2(0)$ to $\gamma_2(t)$ and the latter has a $W_0[f_0]$ -hyperbolic length at most its $B(\gamma_2(0), |\gamma_2(0)|/K)$ -hyperbolic length thus at most $K_5 t$ for T_8 small enough. Now:

$$\Phi_{\text{rep}}[f_y](\gamma_2(y)) = \Phi_{\text{rep}}[f_y](\gamma_0(t)) - (n_1 - n_0).$$

Let us denote $\text{Rep}_y z = \Phi_{\text{rep}}[f_y](z)$. By taking r_0 small enough we can ensure that for all $z \in -D_\theta(r_0)[f_y]$, the quantity $\text{Rep}_y z$ is comparable to $1/z$ and the quantity $\text{Rep}'_y(z)$ is comparable to $1/z^2$ (use the bound on $\tilde{\Phi}$ given in Proposition 7 that extends to W_θ according to Proposition 18). For $y \leq 1/2$, we have $\sup_{|z| < 1/16} |f_0(z) - f_y(z)| \leq Ky$ for some K independent of f . Provided r_2 has been chosen small enough, Proposition 21 gives $|\text{Rep}_y \gamma_2(y) - \text{Rep}_0 \gamma_2(0)| = |\text{Rep}_y \gamma_0(t) - \text{Rep}_0 \gamma_0(t)| = \mathcal{O}(y/|\gamma_0(t)|)$, where Φ_{rep} is normalized by the expansion. Let $u_x = -1/(c[f_0]\gamma_0(x))$ and $Z_x = \text{Rep}_0 \gamma_0(x)$. The size of the quantities $Z_x, Z_0, u_x, u_0, 1/\gamma_0(x)$ and $1/\gamma_0(0)$ are all comparable. Similarly, $|1/\gamma_2(0)|$ is comparable to $|\text{Rep}_0 \gamma_2(0)| = |Z_t - (n_1 - n_0)|$. Note that the positive integer $n_1 - n_0$ can be arbitrarily large. However since Z_t is contained in $-W_{3\pi/4}(10)$ (provided r_2 is small enough), there is an independent lower bound on $|Z_t - (n_1 - n_0)|/|Z_t|$ thus $y/|\gamma_0(t)| = \mathcal{O}(y|Z_0|) = \mathcal{O}(y|Z_t|) \leq \mathcal{O}(y|Z_t - (n_1 - n_0)|) = \mathcal{O}(y|\text{Rep}_0 \gamma_2(0)|) = \mathcal{O}(y/|\gamma_2(0)|)$: for some $M > 0$

$$|\text{Rep}_y \gamma_2(y) - \text{Rep}_0 \gamma_2(0)| \leq My/|\gamma_2(0)|.$$

Then by Proposition 21 we get $|\text{Rep}_y \gamma_2(0) - \text{Rep}_0 \gamma_2(0)| \leq M'y/|\gamma_2(0)|$ thus $|\text{Rep}_y \gamma_2(y) - \text{Rep}_y \gamma_2(0)| \leq |\text{Rep}_y \gamma_2(y) - \text{Rep}_0 \gamma_2(0)| + |\text{Rep}_0 \gamma_2(0) - \text{Rep}_y \gamma_2(0)| \leq (M + M')y/|\gamma_2(0)|$. The straight segment from $\text{Rep}_y \gamma_2(y)$ to $\text{Rep}_y \gamma_2(0)$ is contained in the subset $-W_{\theta'}(R_2)$ of the domain of Rep_y^{-1} and $|(\text{Rep}_y^{-1})'(Z)|$ is comparable to $1/|Z|^2$ for $Z \in -W_{\theta'}(R_2)$. Using moreover that $\text{Rep}_y(Z)$ is comparable to $1/Z$, we get: provided T_8 was chosen small enough, for all $y \leq t$, $|\gamma_2(y) - \gamma_2(0)| \leq K'y|\gamma_2(0)|$. \square

Lemma 58. *If in Proposition 56 we take $n_0 = -\infty$, i.e. start from an orbit indexed by \mathbb{Z} such that $\omega_n\langle 0 \rangle \in D_{\text{rep}}[f_0](r_2)$ for all $n \leq n_1$, and leave the other three assumptions unchanged, then for all $\alpha > 0$ and all $r > 0$, $\exists n' \in \mathbb{Z}$ such that $\forall n \leq n', \forall s \in [0, t], \omega_{n_0}\langle s \rangle$ belongs to the sector of apex 0, radius r , and angle α around the repelling axis of f_s .*

Proof. In the course of the proof of Proposition 56 we proved that $\gamma_0 : s \in [0, t] \mapsto \omega_n\langle s \rangle$ has a support contained in $-D_{\theta'}[f_y](r_0)$ for all $y \leq t$. In particular the function $\chi : s \mapsto -1/c_{f_s}\gamma_0(s)$ takes values in $-W_{\theta'}(1/|c_{f_s}r_0|)$. Recall that on this set, the dynamics differs from the translation by 1 by at most $1/4$. The path χ has compact image. The lemma follows. \square

3.9.6. Bounding the motion of orbits (putting it all together). We now have the tools to prove Proposition 43.

Recall that we are considering an orbit ω_n indexed by \mathbb{Z} of a map $f_0 \in \mathcal{F}$, eventually captured by an attracting petal in the future, by a repelling petal in the past, and defined a movement $\omega_n\langle t \rangle$ of this sequence, for which it remains an orbit of f_t and so that its attracting Fatou coordinate, normalized by immobilizing the image of the critical value, remains constant. The starting hypothesis is that ω_n is entirely contained in $\text{Dom}(f_0) \odot (1 - \varepsilon') = \phi_0(B(0, 1 - \varepsilon'))$. In particular condition (4) of Proposition 50 and its analog in Proposition 51 are satisfied for all $n \in \mathbb{Z}$ by the assumption.

We will now compute a lower bound for the survival time $\tau(\omega_n)$, that depends only on ε' .

This will be done by decreasing induction on n , using Propositions 50, 51 and 56 and their complements Lemmas 52, 53, 54 and 57. The induction hypothesis will be that the motion of $t \mapsto \omega_n \langle t \rangle$, measured with the hyperbolic metric of the set $W_0[f_0]$, more precisely what we called the extent at the beginning of Section 3.9.4, is smaller than the constants d_1 , d'_1 and d''_1 appearing in the propositions. The complements then give an upper bound on the motion of $t \mapsto \omega_{n-1} \langle t \rangle$. We will show that for t small enough, this bound is also less than d_1 , d'_1 and d''_1 , so that the induction can go on, and we will give a lower bound on how small t needs to be.

Recall r_0 is a small enough constant provided by Propositions 7 to 10, and 18.

By Lemma 44, we know the survival of local orbits. More precisely let us choose $T_3 = T'_1/2$. Lemma 44 yields a value r_1 . If the whole orbit $(\omega_n \langle 0 \rangle)_{n \in \mathbb{Z}}$ is contained in $B(0, r_1)$ then we get the lower bound $\tau(\omega_n) \geq T_3$. In this simple case, the lower bound is independent of ε' , so it is even better. In the sequel, we assume that we are not in this case, i.e. that the orbit $(\omega_n \langle 0 \rangle)_{n \in \mathbb{Z}}$ leaves $B(0, r_1)$ at least once.

Recall that maps $f \in \mathcal{F}$ all have the same critical value v . We have already remarked that by compactness of \mathcal{F} and Proposition 11 (see also Lemma 15), there exists η_0 such that $\forall f \in \mathcal{F}$, $B(v, \eta_0)$ is contained in the basin of the parabolic point. Recall $D_{\text{rep}}(r) = D_{\text{rep}}[f](r)$ denotes the disk of diameter $[0, re^{i\theta}]$ where $e^{i\theta}$ points in the direction of the repelling axis of f . Let $f^{\mathbb{N}}(B(v, r))$ denote the union of $B(v, r)$ and of all its images by iteration of f .

Lemma 59. *There exists $r_3 > 0$ and $\eta'_0 < \eta_0$ such that $\forall r \leq r_3$, $\forall f \in \mathcal{F}$, the set $f^{\mathbb{N}}(B(v, \eta'_0))$ is disjoint from $f(B(0, r)) \setminus B(0, r)$ and from $f(D_{\text{rep}}(r))$.*

Proof. Let r_0 be provided by Proposition 7: for all $f \in \mathcal{F}$, and all $r \leq r_0$, $D_{\text{attr}}(r)$ is stable by f and contained in the parabolic basin. Note that for some r' small enough, then for all r small enough, then for all $f \in \mathcal{F}$, $f(B(0, r)) \setminus B(0, r)$ and $f(D_{\text{rep}}(r))$ are disjoint from $D_{\text{attr}}(r')$, as easily follows from uniform bounds on the conjugate of f by the change of variable $u = -1/c_f z$ explained in the proof of Proposition 7: the conjugate differs from the translation by 1 by at most $1/4$, and c_f is bounded away from 0 and from ∞ . By Lemma 15 there is some n_0 and some $\eta'_0 > 0$ such that $\forall f \in \mathcal{F}$, $f^{n_0}(B(v, \eta'_0)) \subset D_{\text{attr}}(r')$, and hence $\forall n \geq n_0$, $f^n(B(v, \eta'_0)) \subset D_{\text{attr}}(r')$. By compactness of \mathcal{F} again, there is a uniform lower bound on the distance from 0 to $f^n(B(v, \eta'_0))$ as n varies between 0 and $n_0 - 1$ and f varies in \mathcal{F} . So the lemma will hold for r small enough. \square

Let T_8 , d'_1 and r_2 be provided by Proposition 56. Let

$$r'_0 = \min(r_0, r_1, r_2, r_3)$$

and denote

$$D_{\text{rep}} = D_{\text{rep}}[f] = D_{\text{rep}}[f](r'_0).$$

We introduced earlier the δ -neighborhood $V_\delta[f]$ of $PC(f)$. Let $\tilde{B}(r) = \tilde{B}(r)[f]$ be the set of points in $B(0, r) \setminus \{0\}$ whose forward orbit by f is contained in $B(0, r)$. Let

$$\tilde{V}_\eta = \tilde{V}_\eta[f] = \tilde{B}(\eta) \cup f^{\mathbb{N}}(B(v, \eta)).$$

By construction, $f(\tilde{V}_\eta) \subset \tilde{V}_\eta$ (do not forget that there is no other preimage of the origin than itself³⁷).

Lemma 60. *The following holds, where $D = D_{\text{rep}}[f](\eta)$:*

$$(1) \quad \forall \eta > 0, \exists \delta > 0 \text{ s.t. } \forall f \in \mathcal{F}, V_\delta[f] \subset \tilde{V}_\eta[f] \cup (D \cap f^{-1}(D)),$$

³⁷And even if there were, it would be sufficient to assume η small enough.

$$(2) \exists \eta_2 > 0, \forall \eta \leq \eta_2, \exists \delta > 0 \text{ s.t. } \forall f \in \mathcal{F}, V_\delta[f] \cap f^{-1}(\tilde{V}_\eta[f]) \subset \tilde{V}_\eta[f],$$

Proof. These are again proved by compactness arguments. Let r_0 be provided by Proposition 7 applied to \mathcal{F} . Then $D_{\text{attr}}[f](r)$ is an attracting petal for all $r \leq r_0$.

- (1) The set $\tilde{B}(\eta) \cup (D_{\text{rep}}[f](\eta) \cap f^{-1}(D_{\text{rep}}[f](\eta))) \subset \tilde{V}_\eta[f] \cup (D_{\text{rep}}[f](\eta) \cap f^{-1}(D_{\text{rep}}[f](\eta)))$ is a neighborhood of 0 thus contains a ball $B(0, r)$. We can take a uniform value of r for maps $f \in \mathcal{F}$ (this can be seen in coordinates $u = -1/c_f z$ as in the proof of Proposition 7: the constant c_f is bounded away from 0 and ∞ and the map f is conjugated to a map $u \mapsto u'$ defined on a uniform neighborhood of ∞ and with $|u' - (u + 1)| < 1/4$). We impose $\delta \leq r/2$. By Lemma 14 for some n_0 we have $\forall f \in \mathcal{F}, f^{n_0}(v_f) \in D_{\text{attr}}[f](r)$ and thus $\forall n \geq n_0, B(f^n(v_f), \delta) \subset \tilde{V}_\eta[f] \cup (D_{\text{rep}}[f](\eta) \cap f^{-1}(D_{\text{rep}}[f](\eta)))$. Finally by compactness there is a lower bound on $\inf \{\delta > 0 \mid \forall f \in \mathcal{F}, \forall k < n_0, B(f^k(v_f), \delta) \subset f^k(B(v_f, \eta))\}$.
- (2) Let $\eta_0 > 0$ to be determined below and set $\eta_2 = \eta_0/2$. Let us assume by contradiction that for some $\eta \leq \eta_0/2$ there exists sequences $\delta_n \rightarrow 0, f_n \in \mathcal{F}, z_n$ such that $z_n \in V_{\delta_n}[f_n], f_n(z_n) \in \tilde{V}_\eta[f_n], z_n \notin \tilde{V}_\eta[f_n]$. We may extract a subsequence so that $f_n \rightarrow f_0$, and $z_n \rightarrow z_0$. If $z_0 \neq 0$ then $z_0 \in PC(f_0)$ (see point (1) of Lemma 16), a fortiori $z_0 \in f_0^{\mathbb{N}}(B(v_{f_0}, \eta))$ and thus for n big enough $z_n \in f_n^{\mathbb{N}}(B(v_{f_n}, \eta))$ by Hurwitz's theorem, thus $z_n \in \tilde{V}_\eta[f_n]$, leading to a contradiction. If $z_n \rightarrow 0$ then for n big enough, let us prove the statement $f_n(z_n) \in \tilde{V}_\eta[f_n] \implies z_n \in \tilde{V}_\eta[f_n]$, which leads to a contradiction. Indeed either $f_n(z_n) \in \tilde{B}(\eta)[f_n]$ but then as soon as $|z_n| < \eta$, the whole orbit of z_n by f_n is in $B(0, \eta)$ and thus $z_n \in \tilde{B}(\eta)[f_n]$ thus $z_n \in \tilde{V}_\eta[f_n]$. Or $f_n(z_n) \in f_n^{\mathbb{N}}(B(v_{f_n}, \eta))$, say $f_n(z_n) \in f_n^{k_n}(B(v_{f_n}, \eta))$. For a fixed k , by compactness there is a lower bound on the distance from 0 to $f^k(B(v_f, \eta_0/2))$ for $k' < k$ and $f \in \mathcal{F}$. So $k_n \rightarrow +\infty$. Now f_n is injective on $B(0, r)$ for some uniform $r \leq r_0$. By Lemma 15 there is some n_0 and $\eta_0 > 0$ such that $\forall f \in \mathcal{F}$, we have $f^{n_0}(B(v_f, \eta_0)) \subset D_{\text{attr}}[f](r)$. As soon as $k_n \geq n_0 + 1$, both $f_n^{k_n-1}(B(v_{f_n}, \eta))$ and $f_n^{k_n}(B(v_{f_n}, \eta))$ are contained in $D_{\text{attr}}[f](r) \subset B(0, r)$, and z_n also belongs to $B(0, r)$ for n big enough. Hence $f(z_n) \in f_n^{k_n}(B(v_{f_n}, \eta)) \implies z_n \in f_n^{k_n-1}(B(v_{f_n}, \eta))$.

□

Let

$$\eta_1 = \min(\eta_0/2, r_0, r_1, r_2, r_3, \delta_1/2, \eta'_0, \eta_2)$$

where δ_1 was defined just before Lemma 49, r_2 in Proposition 56, η_0, r_0 and $r'_0 = \min(r_0, r_1, r_2, r_3)$ at the beginning of the current section (Section 3.9.6), η'_0 and r_3 in Lemma 59, η_2 in Lemma 60.

Let δ be the smallest of the two values associated to $\eta = \eta_1$ by points (1) and (2) of Lemma 60. Since $\eta_1 \leq r'_0$ we get $D_{\text{rep}}[f](\eta_1) \subset D_{\text{rep}}[f](r'_0) = D_{\text{rep}}[f]$ and thus: $\forall f \in \mathcal{F}$,

$$(7) \quad V_\delta[f] \subset \tilde{V}_{\eta_1}[f] \cup (D_{\text{rep}}[f] \cap f^{-1}(D_{\text{rep}}[f])),$$

$$(8) \quad f^{-1}(\tilde{V}_{\eta_1}[f]) \cap V_\delta[f] \subset \tilde{V}_{\eta_1}[f].$$

Let $d_1 = d_1(\delta)$, $d''_1 = d''_1(\delta)$ and $T_4 = T_4(\delta)$ be the values associated to δ just before Proposition 50, and $T_6 = T_6(\delta)$ defined just before Lemma 54.

Just before Proposition 50 we also defined $T_5(\delta, \varepsilon')$, by $\ell(1 - T_5(\delta, \varepsilon')) = d_1(\delta) + \ell(1 - \varepsilon')$ where $\ell(x) = d_{\mathbb{D}}(0, x)$. Since we just have fixed δ , let us denote $T_5(\varepsilon') = T_5(\delta, \varepsilon')$. Then

$$T_5(\varepsilon') \underset{\varepsilon' \rightarrow 0}{\sim} K_3 \varepsilon'$$

with $K_3 = e^{-2d_1(\delta)}$ (the value of this constant is not important, nor is its dependence on δ).

Lemma 61. *There exists $K_2 > 0$ and $T_7 > 0$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \tilde{V}_{\eta_1}[f_0]$, $\tau(z) > T_7$ and for all $t \leq T_7$, the length of the curve $x \in [0, t] \mapsto z\langle x \rangle$ is $\leq K_2 t$ when measured with the hyperbolic metric of $\mathbb{C} \setminus \{v, 0\}$.*

Proof. If the starting point $z\langle 0 \rangle$ belongs to the part $\tilde{B}(\eta_1)$ of \tilde{V}_{η_1} of points whose orbit never leaves $B(0, \eta_1)$, this follows from Lemma 45 since $\eta_1 \leq r_1$ and since $\rho(z) := \rho_{\mathbb{C} \setminus \{0, v\}}(z) = o(1/|z|)$ near 0 thus $\rho(z)|z|$ is bounded on $B(0, \eta_1)$ (note that $\eta_1 < \eta_0 < |v|$). Otherwise the starting point $z\langle 0 \rangle$ belongs to $f_0^{\mathbb{N}}(B(v, \eta_1))$. Note first that only a finite number of iterates of $B(v, \eta_1)$, bounded independently of f_0 , are not already contained in the first part. Moreover, let $m - 1$ be such a bound. Then for all $k \leq m$, for all $z \in f_0^k(B(v, \eta_1))$, $z\langle t \rangle = f_t^{-(m-k)} \circ \Phi_t^{-1} \circ \Phi_0 \circ f_0^{m-k}(z)$ for some inverse branch of $f_t^{(m-k)}$. Since we do not hit a critical point, everything moves differentiably w.r.t. the pair (t, z) . We thus get the claimed bound on the hyperbolic length of the curve $z\langle t \rangle$ away from v , i.e. if $z\langle 0 \rangle \notin B(v, \eta_1)$. Last, for starting points $z\langle 0 \rangle$ near v , i.e. in $B(v, \eta_1)$, note first that v does not move at all: $v\langle t \rangle = v$. Then $|z\langle t \rangle - z| \leq K|z - v|t$ since the function $(z, t) \mapsto z\langle t \rangle - z$ is at least C^2 and vanishes whenever $t = 0$ or $z = v$. Since $\rho(z) = o(1/|z - v|)$ near v , the lemma follows. \square

Recall that we are dealing with the case where the sequence $n \in \mathbb{Z} \mapsto \omega_n\langle 0 \rangle$ is not completely contained in $B(0, r_1)$. Together with Lemma 59 and $\eta_1 \leq r_1$, this implies that the first point in this orbit that does not belong to $B(0, r_1)$ also does not belong to $\tilde{V}_{\eta_1}[f_0]$. On the other hand the orbit tends to 0 thus eventually stays in $B(0, \eta_1)$ hence in $\tilde{B}(\eta_1)[f_0] \subset \tilde{V}_{\eta_1}[f_0]$. The set $\tilde{V}_{\eta_1}[f_0]$ is mapped in itself by f_0 . Therefore there is a unique $n_+ \in \mathbb{Z}$ such that

$$\omega_n\langle 0 \rangle \in \tilde{V}_{\eta_1}[f_0] \iff n \geq n_+.$$

If we follow the orbit in the past, it eventually stays in $D_{\text{rep}} = D_{\text{rep}}[f_0](r'_0)$ in the past. There is thus a maximal $n_- \in \mathbb{Z}$ such that $\forall n \leq n_-$, $\omega_n\langle 0 \rangle \in D_{\text{rep}}$. Moreover, $n_- + 1 < n_+$ because by Lemma 59, $\omega_{n_-+1}\langle 0 \rangle$ cannot belong to $f^{\mathbb{N}}(B(v, \eta_1))$ and if $\omega_{n_-+1}\langle 0 \rangle$ were in $B(0, \eta_1)$ then the whole orbit would be contained in $B(0, r_1)$.

Between n_- and n_+ , the orbit may visit and leave the repelling petal several times. Let J denote the set of $n \in \mathbb{Z}$ with $n_- < n < n_+$ and $\omega_n\langle 0 \rangle \notin D_{\text{rep}}$. This set is non-empty and its extreme values are $n_- + 1$ and $n_+ - 1$ (these two values may be equal).

Denote as follows the constant provided by Lemma 47 and used in Lemma 52:

$$\Lambda := \Lambda(\delta/3) < 1.$$

Let now $t_{\max} \leq \min(T_3, T_4, T_5/2, T_6, T_7, T_8)$ to be determined later. Let us work with $t \in [0, t_{\max}]$ and let us do a finite decreasing induction on J . In the process, more conditions will be imposed on t_0 .

Initialization: By Lemma 61, $\tau(\omega_{n_+}) \geq t_{\max}$ and for all $t \leq t_{\max}$, the length of $\gamma : s \in [0, t] \mapsto \omega_{n_+}\langle s \rangle$ is $\leq K_2 t$ when measured with the hyperbolic metric on $\mathbb{C} \setminus \{0, v\}$. Provided $K_2 t_{\max} \leq d_1''$, we can apply Proposition 51 (in particular condition (5) of this proposition follows from Equation (8)), thus $\tau(\omega_{n_+-1}\langle 0 \rangle) > t_{\max}$ and $\forall t \in [0, t_{\max}]$, the path $s \in [0, t] \mapsto \omega_{n_+-1}\langle s \rangle$ is homotopic in W_0 to $\gamma_1 \cdot \gamma_2$. By Lemma 53, $\text{hlen}_{W_0}(\gamma_1) \leq M(\delta/3) \text{hlen}_{\mathbb{C} \setminus \{0, v\}}(\gamma)$ thus $\leq M_0 K_2 t$ with $M_0 = M(\delta/3)$. By Lemma 54, $\text{hlen}_{W_0}(\gamma_2) \leq K_0 t / T_5$. Finally: we assumed $K_2 t_{\max} \leq d_1''$ and got $\forall t \in [0, t_{\max}]$, $\text{hlen}_{W_0}(\omega_{n_+-1}|_{[0, t]}) \leq M_0 K_2 t + K_0 t / T_5$. In particular

$$\text{extent}_{W_0}(\omega_{n_+-1}\langle [0, t_{\max}] \rangle) \leq M_0 K_2 t_{\max} + K_0 t_{\max} / T_5.$$

Let us assume moreover that

$$M_0 K_2 t_{\max} + K_0 t_{\max} / T_5 \leq \min(d_1, d'_1)$$

so that $\text{extent}_{W_0}(\omega_{n_+-1}\langle[0, t_{\max}]\rangle) \leq \min(d_1, d'_1)$.

Induction: Let $n \in \mathbb{Z}$ satisfying $n_- + 1 < n \leq n_+ - 1$ and either $n \in J$ or $n - 1 \in J$ and assume that we have proved $\tau(\omega_n\langle 0 \rangle) > t_{\max}$ and $\text{extent}_{W_0}(\omega_n\langle[0, t_{\max}]\rangle) \leq \min(d_1, d'_1)$.

By Equation (7), $\omega_{n-1}\langle 0 \rangle \notin V_\delta[f]$ thus condition (5) of Proposition 50 is satisfied. Hence we can apply the proposition and its complement Lemma 54 and we get $\text{hlen}_{W_0}(\omega_{n-1}|_{[0, t]}) \leq \Lambda \min(d_1, d'_1) + K_0 t / T_5$. Let us impose on t_{\max} that $\Lambda \min(d_1, d'_1) + K_0 t_{\max} / T_5 \leq \min(d_1, d'_1)$, so that we get $\text{extent}_{W_0}(\omega_{n-1}\langle[0, t_{\max}]\rangle) \leq \min(d_1, d'_1)$. If $n - 1 \in J$ we can carry on the induction with $n - 1$. If $n - 1 \notin J$, let n' be the first element of J below n and let $n_1 = n - 1$ and $n_0 = n' + 1$: $n_0 \leq n_1$. If $n_0 < n_1$ we can apply Proposition 56 and its complement Lemma 57: $\text{hlen}_{W_0}(\omega_{n_0}|_{[0, t]}) \leq \text{hlen}_{W_0}(\omega_{n_1}|_{[0, t]}) + K_5 t$. We can carry on the induction with n' , provided we require on t_{\max} that $\Lambda \min(d_1, d'_1) + K_0 t_{\max} / T_5 + K_5 t_{\max} \leq \min(d_1, d'_1)$.

In both cases, for the induction to carry on it is enough to assume that

$$\Lambda \min(d_1, d'_1) + K_0 t_{\max} / T_5 + K_5 t_{\max} \leq \min(d_1, d'_1).$$

Post induction: we now know that $\text{extent}_{W_0}(\omega_n\langle[0, t_{\max}]\rangle) \leq \min(d_1, d'_1)$ holds for $n = n_- + 1$. We can apply once more Proposition 56 and we get that the rest of the orbit (for all $n \in \mathbb{Z}$ with $n \leq n_-$) is defined at least up to time t_{\max} . Moreover, by Lemma 58, we get that for all n below some relative integer, possibly much smaller³⁸ than n_- , the full motion takes place in the petal: one of the conclusions of Proposition 43.

Taking everything into account, we get that the full orbit ω_n survives for any time t satisfying $t \leq t_{\max}$ for any t_{\max} satisfying $t_{\max} \leq \min(T_3, T_4, T_5/2, T_6, T_7, T_8)$, $t_{\max} \leq d'_1/K_2$, $t_{\max} \leq \min(d_1, d'_1)/(M_0 K_2 + K_0/T_5)$ and $t_{\max} \leq \min(d_1, d'_1)(1 - \Lambda)/(K_0/T_5 + K_5)$.

Recall that δ is fixed but not ε' . All constants depend only on δ thus are fixed, except, as we saw earlier, $T_5 \sim K_3 \varepsilon'$ (K_3 also depends on δ thus is fixed).

Hence, for ε' small enough, the survival time of the full orbit is $> K_6 \varepsilon'$ for some constant $K_6 > 0$:

$$\boxed{\forall n \in \mathbb{Z}, \tau_\Phi(\omega_n) > K_6 \varepsilon'}.$$

This completes the proof of Proposition 43 with $K = 1/K_6$.

3.10. Step 2, Conclusion. Here we will prove Assertion 42 (which is what is left to prove the main theorem), whose statement we recall:

Assertion. *There exists $r'_0 < r_0$ and a pair $\varepsilon_1 < \varepsilon_0$ with $\varepsilon_0 < T'_1$ such that for all $f_0 \in \mathcal{F}$, for all $z \in \text{Dom}(\mathcal{R}[f_0]) \odot (1 - \varepsilon_1)$, if we consider the orbit ω_n associated to z , then*

- for all $n \in \mathbb{Z}$

$$\tau_\Phi[f_0](\omega_n) > \varepsilon_0,$$

- there exists $M \in \mathbb{Z}$ such that $(t \leq \varepsilon_0 \text{ and } n \leq M) \implies \omega_n\langle t \rangle \in D_{\text{rep}}[f_t](r'_0)$.

Consider $\varepsilon_1 \in]0, 1[$ to be determined later. Let $f_0 \in \mathcal{F}$, and $z \in \text{Dom}(\mathcal{R}[f]) \odot (1 - \varepsilon_1)$ and apply Proposition 22 to $\varepsilon = \varepsilon_1$. For this we have to assume $\varepsilon_1 < \xi$ for some $\xi > 0$ given by the proposition. We obtain some $\varepsilon' = \varepsilon'(\varepsilon_1) > 0$ such that the associated orbit $\omega_n\langle 0 \rangle$ of f_0 is contained in $\text{Dom}(f_0) \odot (1 - \varepsilon')$. By the previous section (Proposition 43), $\forall n \in \mathbb{Z}$, $\tau_\Phi(\omega_n) > \varepsilon'(\varepsilon_1)/K$. We can take $\varepsilon_0 = \varepsilon'(\varepsilon_1)/K$.

³⁸Proposition 43 claims uniformity w.r.t. t , but not w.r.t. f .

For ε_1 small enough, $\varepsilon_0 < T'_1$. Moreover, since $\varepsilon'(\varepsilon) \gg \varepsilon$, for small enough values of ε_1 we have $\varepsilon_0 > \varepsilon_1$. Proposition 43 also provides the second claim in Assertion 42. Q.E.D.

Now comes a final set of remarks. Let us call $(\varepsilon_0, \varepsilon_1)$ a valid pair whenever $\varepsilon_1 < \varepsilon_0 < T'_1$ and the assertion holds with these values. Given ε_1 small enough, the set of valid values for ε_0 includes the interval $]\varepsilon_1, \varepsilon'(\varepsilon_1)/K[$. As the right bound is $\gg \varepsilon_1$, it is easy to deduce that: $\forall \varepsilon_0$ there exists ε_1 such that $(\varepsilon_0, \varepsilon_1)$ is a valid pair. Moreover we can take $\varepsilon_1 = o(\varepsilon_0)$.

This implies that if one iterates renormalization starting from a map in \mathcal{F}_ε with ε small enough, the map $\mathcal{R}^n[f]$ will have at least structure $\mathcal{F}_{\varepsilon_n}$ with $1/\varepsilon_n$ increasing faster than any exponential: the structure tends rapidly to the full structure \mathcal{F} .

Now, given the specific formula in proposition 22:

$$\log \frac{1}{\varepsilon'(\varepsilon_1)} \leq c' + c \log \left(1 + \log \frac{1}{\varepsilon_1} \right)$$

and the computations above, we get that we can take $\varepsilon_1 \leq \exp(\beta - \alpha/\varepsilon_0)$ for some constants $\alpha, \beta > 0$, i.e. $1/\varepsilon_n$ increases at least like an *iterated* exponential.

SUMMARY OF NOTATIONS

$\cdots[f]$	used to emphasize the dependence on f of a given object
A	immediate basin of the parabolic point of $f \in \mathcal{F}$, page 47
B_d	A unicritical Blaschke product with a parabolic point at $z = 1$, page 9
\tilde{B}_d	Another normalization of B_d , page 19
b_*	the cubox that contains a punctured neighborhood of the origin, page 54
β_t	constant so that $\Phi_t(z) = \Phi_{\text{attr}}[f_t](z) + \beta_t$ has a critical value independent of t ; $\beta_t = \Phi_{\text{attr}}[f_0](v_0) - \Phi_{\text{attr}}[f_t](v_t)$, page 63
\mathcal{C}	A curve through the orbit of the critical value, Proposition 13, page 43
C	main object of study of Section 3.6, page 48
C_d	A semiconjugate of B_d , so that the parabolic point has only one attracting petal, page 19
\mathbb{D}	the open unit disk in \mathbb{C}
d_1	infimum over \mathcal{F} of some hyperbolic distance, page 75
$\text{Dom}(f)$	domain of definition of the map f
d_U	hyperbolic distance w.r.t. U , page 33
f_0	an element of \mathcal{F} , page 61
\mathcal{F}	Shishikura's invariant class, page 33
\mathcal{F}_ε	a class of maps with slightly less structure, page 33
f_t	a deformation of f_0 , element of \mathcal{F}_t , page 61
\mathbb{H}	the upper half plane in \mathbb{C}
h_{nor}	normalized extended horn maps, $h_{\text{nor}} = \Phi_{\text{attr}} \circ \Psi_{\text{rep}}$, page 35
$\ell(x)$	the hyperbolic distance from 0 to x in \mathbb{D} , page 75
$\lambda[f_0](z)$	some contraction factor in W_0 , page 70
\odot	$U \odot r$ is the set of points $z \in U$ with $d_U(0, z) < d_{\mathbb{D}}(0, r)$, page 33
\Vdash	$V \Vdash r$ is the set of points $z \in V$ with $E(z) \in E(V) \odot r$, page 33
$PC(f)$	the post critical set of f
\mathcal{R}	the upper parabolic renormalization, page 5
r_2	defined in Proposition 56
ρ_U	element of hyperbolic metric w.r.t. U , page 33
\mathcal{S}	The class of Schlicht maps, page 2
σ_t	a motion appearing in the decomposition $f_t = f_0 \circ \sigma_t$, page 72
T_0	for $f \in \mathcal{F}_{[0, T_0[}$ have a (unique) critical value, page 62
T_1	$z \mapsto z + 1$

T'_1	for $f \in \mathcal{F}_{[0, T'_1]}$, the critical value is attracted to 0, page 62
T_3	some parameter in Lemma 44, later chosen to be $= T'_1/2$, page 80
T_5	$\exists! T_5 \in]0, 1[$ s.t. $\ell(1 - T_5) = d_1(\delta) + \ell(1 - \varepsilon')$, page 75
U_1	domain of $f \in \mathcal{F}$, page 47
U_u	upper component of $\Psi_{\text{rep}}^{-1}(A)$, also of $\text{Dom}(h_{\text{nor}})$, page 48
$V_\delta[f]$	the δ -neighborhood of $PC(f)$
$\tilde{V}_\eta[f]$	some domain used in the proofs, page 80
W_0	the complement in \mathbb{C} of the closure of the post critical set of f_0 , page 70
W'_0	$W'_0 = f_0^{-1}(W_0)$, page 70
$W_\theta(R)$	some domain in the coordinates $u = -1/cz$, extending the half plane on which we control the Fatou coordinates, page 45
Φ_{attr}	Attracting Fatou coordinates. Normalized and extended except at the beginning of Section 1.2. Normalized by the expansion at infinity in Section 3.
Ψ_{rep}	Repelling inverse Fatou coordinates. Same remarks as for Φ_{attr} apply.
Φ_t	$\Phi_t = \Phi_{\text{attr}}[f_t] + \beta_t$ with β_t a constant so that the critical value of Φ_t is independent of t , page 63
Ψ_t	$\Psi_t(z) = \Psi_{\text{rep}}[f_t](z - \beta'_t)$ for $\beta'_t = \beta_t - i\pi\gamma[f_t]$ with γ the iterative residue, page 63

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